Math 323
Linear Algebra and Matrix Theory I
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Key Homework 13

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a) True!
b) False!
c) True!
d) False!

Strang Page 5 no: 2

```
A=[1 3 0 2 -1; 0 0 1 4 -3; 1 3 1 6 -4], R=rref(A)

A =
     1     3     0     2    -1
     0     0     1     4    -3
     1     3     1     6    -4
R =
     1     3     0     2    -1
     0     0     1     4    -3
     0     0     0     0     0
```

Let E2inv denote the new E matrix consisting of the first two columns of E, while R2 consists of the first two rows of R.

```
E2inv=[A(:, 1) A(:, 3)], R2=R(1:2, : )

E2inv =
     1     0
     0     1
     1     1
R2 =
     1     3     0     2    -1
     0     0     1     4    -3
```

We check the result.

```
A, E2InvR2=E2inv*R2

A =
     1     3     0     2    -1
     0     0     1     4    -3
     1     3     1     6    -4
```

```
E2InvR2 =
     1     3     0     2    -1
     0     0     1     4    -3
     1     3     1     6    -4
```

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Observe that reordering the rows of a matrix does not change the dependency relations among its columns. This implies that the Reduced Row Echelon form of a matrix is independent of the order of the rows. Reorder the rows of A to obtain a matrix B such that the first k rows of B are linearly independent and the last $n-k$ rows of B are linear combinations of the first k rows of B. Computing the reduced row echelon form of B must result in a matrix with k non-zero rows. Since rref(B) = rref(A) this implies that $k = r = rank(A)$. We now know that the rank of A is not just the number of linearly independent columns of A, it is also the number of linearly independent rows of A. The latter gives us that A and A^T have the same rank r, and therefore they have the same number of pivot columns even though the column numbers can be different.

As an example take:

```
A=[0 0 0; 0 0 0; 0 1 0], rrefA=rref(A), rrefAT=rref(A')

A =
     0     0     0
     0     0     0
     0     1     0
rrefA =
     0     1     0
     0     0     0
     0     0     0
rrefAT =
     0     0     1
     0     0     0
     0     0     0
```

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a)

```
R=[1 0 2 3; 0 1 4 5; 0 0 0 0], x=nulbasis(R ), y=nulbasis(R')

R =
     1     0     2     3
     0     1     4     5
     0     0     0     0
x =
    -2    -3
    -4    -5
     1     0
     0     1
y =
     0
     0
     1
```

b)

```
R=[0 1 2; 0 0 0; 0 0 0], x=nulbasis(R ), y=nulbasis(R')

R =
     0     1     2
     0     0     0
     0     0     0
x =
     1     0
     0    -2
     0     1
y =
     0     0
     1     0
     0     1
```

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a)

```
A=[1 2 3; 1 2 4], rrefA=rref(A)

A =
     1     2     3
     1     2     4
rrefA =
     1     2     0
     0     0     1

answer=[A(:, 1) A(:, 3)]

answer =
     1     3
     1     4
```

b)

```
A=[1 2 3; 2 4 6], rrefA=rref(A)

A =
     1     2     3
     2     4     6
rrefA =
     1     2     3
     0     0     0

answer=[A(1, 1)]

answer =
     1
```

c)

```
A=[0 1 0; 0 0 1; 0 0 0], rrefA=rref(A)

A =
     0     1     0
     0     0     1
     0     0     0
```

```
rrefA =
     0     1     0
     0     0     1
     0     0     0
```

```
answer=[A(1:2, 2:3)]
```

```
answer =
     1     0
     0     1
```

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Gauss-Jordan elimination applied to this new matrix produces an m by r matrix P, whose first r rows form the r by r identity matrix, while the last m - r rows are zero rows. Clearly the rank of P equals r.

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In problem 7 we proved that the rank of a matrix equals the rank of its transpose, $rank(P^T) = rank(P) = r$.

```
A=[1 2 3; 2 4 6; 2 4 7], rrefA=rref(A)
```

```
A =
     1     2     3
     2     4     6
     2     4     7
rrefA =
     1     2     0
     0     0     1
     0     0     0
```

```
P=[A(:, 1) A(:,3)], PT=P', rrefPT=rref(P')
```

```
P =
     1     3
     2     6
     2     7
PT =
     1     2     2
     3     6     7
rrefPT =
     1     2     0
     0     0     1
```

```
ST=[PT(:,1), PT(:, 3)], S=ST'
```

```
ST =
     1     2
     3     7
S =
     1     3
     2     7
```

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a)
A column of AB equals A times the corresponding column of B. Therefore, if a column of B happens to be a linear combination of the previous columns of B, then the same column of AB will be the same linear combination of the previous columns of AB. In formula: If $\vec{b}_k = \alpha_{k-1}\vec{b}_{k-1} + \cdots + \alpha_1\vec{b}_1$, then

$$A\,\vec{b}_k = A\,(\alpha_{k-1}\vec{b}_{k-1} + \cdots + \alpha_1\vec{b}_1) = \alpha_{k-1}A\,\vec{b}_{k-1} + \cdots + \alpha_1 A\,\vec{b}_1.$$

b)
Observe that the rank of B equals 1. It is trivially true that the rank of $I\,B$ equals the rank of B. The next lower rank number is zero. Only the zero matrix has rank zero. $A\,B = 0$, for instance if :

```
B=[1 2; 3 6], A=[-3 1; -3 1], AB=A*B, rankAB=rank(AB)
```

```
B =
     1     2
     3     6
A =
    -3     1
    -3     1
AB =
     0     0
     0     0
rankAB =
     0
```

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$rank(B^T A^T) \le rank(A^T) \Rightarrow rank(A\ B) \le rank(A)$.

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If A and B are n by n matrices and $A\ B = I$, then $n = rank(I) = rank(A\ B) \le rank(A) \le n$, which implies that $rank(A) = n$, so A is invertible and therefore $B = A^{-1}$ and $B\ A = I$.

This is very quick proof of the fact that if a square matrix has a right inverse, then it is invertible. Similarly if, we focus on B rather than on A, we immediately deduce that if a square matrix has a left inverse, then it is invertible.

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a)

```
A=[1 1 0; 1 1 4; 1 1 8], B=[A A], rrefA=rref(A), rrefB=rref(B)
```

```
A =
     1     1     0
     1     1     4
     1     1     8
B =
     1     1     0     1     1     0
     1     1     4     1     1     4
     1     1     8     1     1     8
rrefA =
     1     1     0
     0     0     1
     0     0     0
rrefB =
     1     1     0     1     1     0
     0     0     1     0     0     1
     0     0     0     0     0     0
```

EA=[A(:, 1) A(:, 3)], EB=EA

```
EA =
     1     0
     1     4
     1     8
EB =
     1     0
     1     4
     1     8
```

S1=EA(:, 1)*rrefA(1, :), S2=EA(:, 2)*rrefA(2, :), A, check=S1+S2

```
S1 =
     1     1     0
     1     1     0
     1     1     0
S2 =
     0     0     0
     0     0     4
     0     0     8
A =
     1     1     0
     1     1     4
     1     1     8
check =
     1     1     0
     1     1     4
     1     1     8
```

b)

T1=EB(:, 1)*rrefB(1, :), T2=EA(:, 2)*rrefB(2, :), B, check=T1+T2

```
T1 =
     1     1     0     1     1     0
     1     1     0     1     1     0
     1     1     0     1     1     0
T2 =
     0     0     0     0     0     0
     0     0     4     0     0     4
     0     0     8     0     0     8
```

```
B =
     1     1     0     1     1     0
     1     1     4     1     1     4
     1     1     8     1     1     8
check =
     1     1     0     1     1     0
     1     1     4     1     1     4
     1     1     8     1     1     8
```