Math 323
Linear Algebra and Matrix Theory I
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Key Homework 14

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```
A=[1 3 3 ; 2 6 9; -1 -3 3], b=[1 5 5]'

A =
     1     3     3
     2     6     9
    -1    -3     3
b =
     1
     5
     5

xp=partic(A, b), s=nulbasis(A)

xp =
    -2
     0
     1
s =
    -3
     1
     0
```

The complete solution is given as $\vec{x} = \vec{x}_p + \alpha\,\vec{s}$.

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a) Unfortunately our version of MATLAB does not handle symbolics. The easiest way to solve this problem is to find a matrix E with the property that EA = R, where R denotes the reduced row echelon form of A. Clearly the first two columns of E^{-1} are the pivot colums of A. we augment them to find a basis for R^4.

```
format short
A=[1 2; 2 4; 2 5; 3 9], R=rref(A)

A =
     1     2
     2     4
     2     5
     3     9
```

```
R =
     1     0
     0     1
     0     0
     0     0
```

```
Ah=[A, eye(4)]
```

```
Ah =
     1     2     1     0     0     0
     2     4     0     1     0     0
     2     5     0     0     1     0
     3     9     0     0     0     1
```

```
RAh=rref([A, eye(4)])
```

```
RAh =
    1.0000         0         0         0    3.0000   -1.6667
         0    1.0000         0         0   -1.0000    0.6667
         0         0    1.0000         0   -1.0000    0.3333
         0         0         0    1.0000   -2.0000    0.6667
```

```
format rat
Einv=Ah(:, 1:4), E=inv(Einv)
```

```
Einv =
      1            2            1            0
      2            4            0            1
      2            5            0            0
      3            9            0            0
E =
      0            0            3           -5/3
      0            0           -1            2/3
      1            0           -1            1/3
      0            1           -2            2/3
```

Observe that EA = R

```
E*A, R
```

```
ans =
      1            0
      0            1
      0            0
      0            0
R =
      1            0
      0            1
      0            0
      0            0
```

this means that the desired equations can be read from the third and fourth rows of E:

$b_1 - b_3 + \frac{1}{3}b_4 = 0 \Rightarrow 3b_1 - 3b_3 + b_4 = 0$ and $b_2 - 2\,b_3 + \frac{2}{3}b_4 = 0 \Rightarrow 3b_2 - 6\,b_3 + 2b_4 = 0$.

The solutions for $\vec{x}$ in that case are $\vec{x}$ = [$\begin{matrix} 3b_3 - \frac{5}{3}b_4 \\ -b_3 + \frac{2}{3}b_4 \end{matrix}$] Observe there is just one solution (no free variables, A has full column rank, the nullspace of A consists of the zero vector).

b) The solution is analogous to that of a).

```
format short

A=[1 2 3; 2 4 6; 2 5 7; 3 9 12], R=rref(A)

A =
     1     2     3
     2     4     6
     2     5     7
     3     9    12
R =
     1     0     1
     0     1     1
     0     0     0
     0     0     0

format short
Ah=[A(:, 1:2), eye(4)]

Ah =
     1     2     1     0     0     0
     2     4     0     1     0     0
     2     5     0     0     1     0
     3     9     0     0     0     1

RAh=rref([Ah, eye(4)])

RAh =
  Columns 1 through 7
    1.0000         0         0         0    3.0000   -1.6667         0
         0    1.0000         0         0   -1.0000    0.6667         0
         0         0    1.0000         0   -1.0000    0.3333    1.0000
         0         0         0    1.0000   -2.0000    0.6667         0
  Columns 8 through 10
         0    3.0000   -1.6667
         0   -1.0000    0.6667
         0   -1.0000    0.3333
    1.0000   -2.0000    0.6667


format rat
Einv=Ah(:, 1:4), E=inv(Einv)
```

```
Einv =
      1            2            1            0
      2            4            0            1
      2            5            0            0
      3            9            0            0
E =
      0            0            3         -5/3
      0            0           -1          2/3
      1            0           -1          1/3
      0            1           -2          2/3
```

this means that the desired equations can again be read from the third and fourth rows of E: $b_1 - b_3 + \frac{1}{3}b_4 = 0$ and $b_2 - 2b_3 + \frac{2}{3}b_4 = 0$. The solutions for $\vec{x}$ in that case are:

$$\vec{x} = \begin{bmatrix} 3\,b_3 - \frac{5}{3}b_4 \\ -b_3 + \frac{2}{3}b_4 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Observe there are infinitely many solutions, since $\vec{x}_3$ is a free variable.

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A 1 by 3 system has at least two free variables, so we need two special solutions to A x = 0.

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a) $\vec{x} = \vec{x}_1 - \vec{x}_2$, $\vec{y} = 4\vec{x}_1 - 4\vec{x}_2$

b) $\vec{z} = 3\vec{x}_1 - 3\vec{x}_2$ is another solution to A x = 0,
and $\vec{w} = 4\vec{x}_1 - 3\vec{x}_2$ is another solution to A x = b.

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a) If $\vec{b} \neq \vec{0}$, then $A\,(2\vec{x}_p) \neq \vec{b}$.

b) If A has free columns, and $A\,\vec{x} = \vec{b}$ is solvable, then it has infinitely many solutions.

c) If A and $\vec{b}$ are given by:

```
format short
A=[1 1; 0 0], b=[5, 0]'

A =
     1     1
     0     0
b =
     5
     0
```

then the solution with the free variable x_2 equal to zero is x = (5, 0). However y = (2,3) is also a solution and the length of y is less than the length of x.

```
length_of_y=norm([2 3]', 2), length_of_x=norm([5 0]', 2)

length_of_y =
    3.6056
length_of_x =
     5
```

d) There is always the homogeneous solution $\vec{x}_n = \vec{0}$.

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Then that row is a **zero** row. The equation U x = c is only solvable provided $c_3 = 0$. The equation A x = b **might not be** solvable.

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The largest possible rank of a 6 by 4 matrix is **4**. Then there is a pivot in every **column** of U. The solution to A x = b is **unique**. The nullspace of A is $\{\vec{0}\}$. An example is:

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

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a)

```
A=[1 4 0; 2 11 5; -1 2 10], R=rref(A)

A =
     1     4     0
     2    11     5
    -1     2    10
R =
    1.0000         0   -6.6667
         0    1.0000    1.6667
         0         0         0
```

The rank of A equals 2.

b) Clearly the rank of A equals at least 2. To determine when rank(A) = 2 and when rank(A) = 3, we again use the E matrix.

```
Ah=[[1 0; 1 1; 1 1] eye(3)], RAh=rref(Ah)

Ah =
     1     0     1     0     0
     1     1     0     1     0
     1     1     0     0     1
RAh =
     1     0     1     0     0
     0     1    -1     0     1
     0     0     0     1    -1
```

```
Einv=Ah(:, [1 2 4]), E=inv(Einv)

Einv =
     1     0     0
     1     1     1
     1     1     0
E =
     1     0     0
    -1     0     1
     0     1    -1
```

From the last row of E we deduce that the rank of A equals 2, if $2-q=0 \Rightarrow q=2$ and the rank of A equals 3 if $q \neq 2$.

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We can manually find the L matrix by augmenting A with the identity matrix and applying Gaussian elimination.

```
A=[3 4 1 0; 6 5 2 1], aug=[A, eye(2)]

A =
     3     4     1     0
     6     5     2     1
aug =
     3     4     1     0     1     0
     6     5     2     1     0     1
```

```
A1=rowcomb(aug, 1, 2, -2)

A1 =
     3     4     1     0     1     0
     0    -3     0     1    -2     1
```

The L matrix is now the inverse of the matrix formed by the last two columns of A1.

```
L=inv(A1(:, 5:6)), U=A1(:, 1:4), A, LU=L*U
```

```
L =
     1     0
     2     1
U =
     3     4     1     0
     0    -3     0     1
A =
     3     4     1     0
     6     5     2     1
LU =
     3     4     1     0
     6     5     2     1
```

Alternatively, we can find L by applying LU decomposition to any extension of the pivot columns of A to a basis fo R^m.

a)

```
A=[3 4 1 0; 6 5 2 1], R=rref(A)

A =
     3     4     1     0
     6     5     2     1
R =
    1.0000         0    0.3333    0.4444
         0    1.0000         0   -0.3333

[L, u]=slu(A(:, 1:2))

L =
     1     0
     2     1
u =
     3     4
     0    -3
```

Observe that $A = L\,U$ (capital U, not u) $\Rightarrow U = L^{-1}\ A$.

```
U=inv(L)*A

U =
     3     4     1     0
     0    -3     0     1

L, U, A, LU=L*U

L =
     1     0
     2     1
U =
     3     4     1     0
     0    -3     0     1
A =
     3     4     1     0
     6     5     2     1
```

```
LU =
     3     4     1     0
     6     5     2     1
```

b)

This problem is similar to a)

```
A=[1 0 1 0; 2 2 0 3; 0 6 5 4], aug=[A eye(3)]

A =
     1     0     1     0
     2     2     0     3
     0     6     5     4
aug =
     1     0     1     0     1     0     0
     2     2     0     3     0     1     0
     0     6     5     4     0     0     1

A1=rowcomb(aug, 1, 2, -2)

A1 =
     1     0     1     0     1     0     0
     0     2    -2     3    -2     1     0
     0     6     5     4     0     0     1

A2=rowcomb(A1, 2, 3, -3)

A2 =
     1     0     1     0     1     0     0
     0     2    -2     3    -2     1     0
     0     0    11    -5     6    -3     1


L=inv(A2(:, 5:7)), U=A2(:, 1:4), A, LU=L*U

L =
     1     0     0
     2     1     0
     0     3     1
U =
     1     0     1     0
     0     2    -2     3
     0     0    11    -5
A =
     1     0     1     0
     2     2     0     3
     0     6     5     4
LU =
     1     0     1     0
     2     2     0     3
     0     6     5     4
```

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a)
Clearly the complete solution is given by
$\vec{x} = \vec{x}_p + \alpha\, \vec{x}_{n_1} + \beta\, \vec{x}_{n_2} = (4,0,0) + \alpha\,(-1,1,0) + \beta\,(-1,0,1)$.

b) We use MATLAB.

```
A=[1 1 1; 1 -1 1], b=[4 4]', xp=partic(A, b), xn=nulbasis(A)

A =
     1     1     1
     1    -1     1
b =
     4
     4
xp =
     4
     0
     0
xn =
    -1
     0
     1
```

Clearly the complete solution is given by $\vec{x} = \vec{x}_p + \alpha\, \vec{x}_n$.

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a)

```
u=[3 1 4]', v=[1 2 2]', uvT=u*v'

u =
     3
     1
     4
v =
     1
     2
     2
uvT =
     3     6     6
     1     2     2
     4     8     8
```

b)

```
u=[2 -1]', v=[1 1 3 2]', uvT=u*v'

u =
     2
    -1
```

```
v =
     1
     1
     3
     2
uvT =
     2     2     6     4
    -1    -1    -3    -2
```

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a) Let A have full column rank, but not full row rank.

1	0
0	1
0	0

b) Let A have full row rank, but not full column rank.

1	0	0
0	1	0

c) Let A have neither full row rank, nor full column rank.

1	0	0
0	1	0
0	0	0

d) Let A have full row rank and full column rank (in which case A is invertible)

1	0
0	1

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```
A=[1 0 2 3; 1 3 2 0; 2 0 4 9], b=[2 5 10]'

A =
     1     0     2     3
     1     3     2     0
     2     0     4     9
b =
     2
     5
    10

R=rref([A b])
```

```
R =
     1     0     2     0    -4
     0     1     0     0     3
     0     0     0     1     2
```

Clearly xp = (-4, 3, 0, 2) and the special solution to A x = 0 is s = (-2, 0, 1, 0). All homogeneous solutions are given by c (-2, 0, 1, 0). These results are readily verified using the partic and nulbasis commands.

`xp=partic(A, b), s=nulbasis(A)`

```
xp =
    -4
     3
     0
     2
s =
    -2
     0
     1
     0
```