Math 323
Linear Algebra and Matrix Theory I
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Key Homework 18

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a) dim(C(A)) = dim(R(A)) = 5, dim(N(A)) = 9 - 5 = 4, dim(left nullspace of A) = 7 - 5 = 2.
b) The column space of this matrix is R^3 and the left nullspace contains only the zero vector.

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a)
```
A=[1 0; 1 0; 0 1]

A =
     1     0
     1     0
     0     1
```

b) This is not possible, because if the desired matrix A is, m by n, with rank r, then
- Having one vector as a basis for the column space means r = 1.
- Having one vector in R^3 as a basis for the nullspace implies that n = 3, while dim(N(A)) = 1

These items contradict the identity dim(N(A)) = n - r.

c)
```
A=randint(3, 4, 5, 2), dimension_null_space=rank(nulbasis(A)),
dimension_left_null_space=rank(nulbasis(A'))

A =
    -1     3     2    -1
     4     2    -1     4
    -1     1     1    -1
dimension_null_space =
     2
dimension_left_null_space =
     1
```
d)
```
v=[3 1]', A=[-3*v v]'

v =
     3
     1
A =
    -9    -3
     3     1
```

e) This is not possible. If the row space of A equals the column space of A, then A is square. Say A is n by n. We have shown in class that every vector in N(A) is orthogonal to every vector in R(A). The converse is also true, every vector that is orthogonal to all the vectors of R(A) is an element of N(A). This means that N(A) is exactly the set of all vectors in R^n, which are perpendicular to all vectors in R(A). Similarly the left nullspace of A is exactly the set of all vectors in R^n, which are perpendicular to all vectors in C(A) = R(A). Therefore the nullspace of A and the left nullspace of A must be equal.

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```
A=[1 1 1; 2 1 0], B=[1 -2 1]

A =
     1     1     1
     2     1     0
B =
     1    -2     1
```

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Dim(R(A)) = dim(C(A)) = 3, dim(N(A) = 2, dim(LN(A)) = 0
Dim(R(B)) = dim(C(B)) = 3, dim(N(B) = 3, dim(LN(B)) = 2
Dim(R(C)) = dim(C(C)) = 0, dim(N(A) = 2, dim(LN(A)) = 3

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We need a, 3 by 3, matrix A such that the columns of A are linear combinations of the vectors (1, 0, 1) and (1, 2, 0). This implies that A must be of the form BX, where

```
B=[1 1; 0 2; 1 0]

B =
     1     1
     0     2
     1     0
```

Similar, since the rows of A are linear combinations of the vectors
(1, 0, 1) and (1, 2, 0), A must be of the form YC, where

```
C=B'

C =
     1     0     1
     1     2     0
```

Now simply take X = C and Y = B, then

```
A=B*C

A =
     2     2     1
     2     4     0
     1     0     1
```

It is not possible to make these vectors a basis for the row space and the nullspace, because this would require that 2 + 2 = dim(R(A)) +dim(N(A)) = dim(C(A)) + dim(N(A)) = 3.

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Let A = BC. The rows of A are linear combinations of the rows of C $\Rightarrow R(A) \subseteq R(C)$. Moreover the matrix B is invertible so $B^{-1}A = C$, so the rows of C are linear combinations of the rows of A $\Rightarrow R(C) \subseteq R(A)$. Together we find that A and C have equal rowspaces and since the rows of C are linearly independent, they form a basis for the rowspace of A.

A similar argument shows that R^3= C(B) = the space spanned by the first three columns of A $\Rightarrow C(A) = R^3$ and a basis for C(A) is given by the columns of the, 3 by 3, identity matrix.

Again, since B is invertible: $N(A) = \{\vec{x} | A\,\vec{x} = \vec{0}\} = \{\vec{x} | BC\,\vec{x} = \vec{0}\} = \{\vec{x} | C\,\vec{x} = \vec{0}\} = N(C)$. Thus a basis for N(A) is given by {(0, 1,-2 1)}

Finally, since A has full row rank, the transpose of A must have full column rank. This means that the left nullspace of A, which equals the nullspace of the transpose of A, must consist of only the zero vector. So the basis of the left nullspace is empty.

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The row space and the nullspace stay the same, because changing the order of the row vectors does neither affect the span of the row vectors, nor the dependency relations between the column vectors.

If (1, 2, 3, 4) is in the column space of A, then (2, 1, 3, 4) is in the column space of the new matrix.

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a) The easiest way to solve this problem is to use the PLU decomposition of the matrix

```
A=[1 2; 3 4; 4 6], [P, L, U]=plu(A)

A =
     1     2
     3     4
     4     6
Pivots in columns:
     1     2
Pivots in rows:
     1     2
```

```
P =
     1     0     0
     0     1     0
     0     0     1
L =
     1     0     0
     3     1     0
     4     1     1
U =
     1     2
     0    -2
     0     0
```

Since $PA = LU$, $L^{-1}PA = U$, So the coefficients $\vec{c}$ of linear combination of the rows of A which produces the zero vector, are given by the third row of the $L^{-1}P$ matrix.

```
H=inv(L)*P;, c=H(3, : )

c =
    -1    -1     1
```

This vector $\vec{c}$ forms a basis for the left nullspace of A.

```
check=nulbasis(A')

check =
    -1
    -1
     1
```

b) This problem is similar to that in 19a, but the dimension of the left nullspace is higher than one. Observe.

```
B=[1 2; 2 3; 2 4; 2 5], [P, L, U]=plu(B)

B =
     1     2
     2     3
     2     4
     2     5
Pivots in columns:
     1     2
Pivots in rows:
     1     2
P =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
L =
     1     0     0     0
     2     1     0     0
     2     0     1     0
     2    -1     0     1
```

```
U =
     1     2
     0    -1
     0     0
     0     0
```

A basis for the left nullspace of B is now given by the last two rows of the matrix $L^{-1}P$.

```
H=inv(L)*P; Basis_Left_Nullspace=H(3:4, : )

Basis_Left_Nullspace =
    -2     0     1     0
    -4     1     0     1
```

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a) Since every column of A consists of a linear combination of the vectors $\vec{u}$ and $\vec{w}$, they must span the column space of A.

b) Similarly $\vec{v}^T$ and $\vec{z}^T$ span the row space of A.

c) … if $\vec{u}$ and $\vec{w}$ are linearly dependent, or if $\vec{v}$ and $\vec{z}$ are linearly dependent .

```
u=[1 0 0]'; z=u; v=[0 0 1]'; w=v; A=u*v'+w*z', rank(A)

A =
     0     0     1
     0     0     0
     1     0     0
ans =
     2
```

ATLAST Page 87 no: 7

a) To save some typing we will use MATLAB to column reduce this matrix.

```
format rat
A=[3 1 1 0; -1 -1 1 -2; 2 1 0 1]

A =
      3             1             1             0
     -1            -1             1            -2
      2             1             0             1

A2=rowcomb(A', 1, 2, -1/3)'

A2 =
      3             0             1             0
     -1          -2/3             1            -2
      2           1/3             0             1

A3=rowcomb(A2', 1, 3, -1/3)'
```

```
A3 =
      3              0              0              0
     -1             -2/3           4/3            -2
      2              1/3           -2/3            1
```

A4=rowcomb(A3', 2, 3, 2)'

```
A4 =
      3              0              0              0
     -1             -2/3            *             -2
      2              1/3            *              1
```

A5=rowcomb(A4', 2, 4, -3)'

```
A5 =
      3              0              0              0
     -1             -2/3            *              0
      2              1/3            *              0
```

This is the column echelon form of the matrix A. Three additional steps will bring us to the reduced column echelon form.

A6=rowcomb(A5', 2, 1, -3/2)'

```
A6 =
      3              0              0              0
      0             -2/3            *              0
     3/2             1/3            *              0
```

A7=rowscale(A6', 1, 1/3)'

```
A7 =
      1              0              0              0
      0             -2/3            *              0
     1/2             1/3            *              0
```

A8=rowscale(A7', 2, -3/2)'

```
A8 =
      1              0              0              0
      0              1              *              0
     1/2            -1/2            *              0
```

This is the reduced column echelon form of the matrix A.

b)
B=A8, check=nulbasis(B(:, 1:2))

```
B =
      1              0              0              0
      0              1              *              0
     1/2            -1/2            *              0
check =
   Empty matrix: 2-by-0
```

It is clear that the noin zero columns of B are linearly independent and since column reduction does not affect the column space, these non zero columns of B for a basis for the column space of A.

c)

```
A=[2 -1 3 1 -1; 0 0 -1 0 1; 1 -2 -1 1 2; 2 2 1 0 1], B=rref(A')'

A =
     2    -1     3     1    -1
     0     0    -1     0     1
     1    -2    -1     1     2
     2     2     1     0     1
B =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     2     7    -2     0     0

basis_for_the_column_space_of_A=B(:, 1:3)

basis_for_the_column_space_of_A =
     1     0     0
     0     1     0
     0     0     1
     2     7    -2
```

d) We compute the matrix Q such that $AQ = B$.

```
H=rref([A;eye(5)]')', Q=H(5:9, : )

H =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     2     7    -2     0     0
     0     0     0     1     0
     0     0     0     0     1
     1     3    -1    -1    -1
    -1    -5     2     0     3
     1     4    -1    -1    -1
Q =
     0     0     0     1     0
     0     0     0     0     1
     1     3    -1    -1    -1
    -1    -5     2     0     3
     1     4    -1    -1    -1
```

e)

```
B, Q

B =
     1     0     0     0     0
     0     1     0     0     0
     0     0     1     0     0
     2     7    -2     0     0
```

```
Q =
     0           0           0           1           0
     0           0           0           0           1
     1           3          -1          -1          -1
    -1          -5           2           0           3
     1           4          -1          -1          -1
```

```
Basis_for_the_nullspace_of_A=Q(:, 4:5)
```

```
Basis_for_the_nullspace_of_A =
     1           0
     0           1
    -1          -1
     0           3
    -1          -1
```

- These two vectors are indeed in the nullspace of A because they satify the equation $A\,\vec{x} = \vec{0}$.
- These two vectors are also linearly independent bacause they are columns of an invertible matrix. Recall that Q is invertible because it is a product of elimination matrices.

f) In light of the argument made in (e), all we have to do to complete the argument, is to show that the dimensions are right. Since B and A have equal column spaces, they have equal ranks. If rank(A) = rank(B) = r, the number of zero columns in B equals n - r. This implies that this new procedure will always produce a set of n - r linearly independent vectors in the nullspace of A. But since the nullspace of A has dimension n - r, we actually produced a basis for that nullspace.

g) Observe using block matrices that if $[A\,\vec{b}][\begin{smallmatrix}\vec{x}\\ -1\end{smallmatrix}] = A\,\vec{x} - \vec{b} = \vec{0}$, we automatically have a solution to $A\,\vec{x} = \vec{b}$. If $\vec{q}$ denotes any column of Q such that $[A\,\vec{b}]\,\vec{q} = \vec{0}$, then $[A\,\vec{b}]\,c\,\vec{q} = \vec{0}$ for any scalar c. All we have to do to fand a solution to $A\,\vec{x} = \vec{b}$, is to devide by the negative of their last component, the columns of Q which generate the zero columns in the reduced column echelon form of $[A\,\vec{b}]$.

Let's do it !

```
format short
b1=[-1 2 3 6]'
```

```
b1 =
    -1
     2
     3
     6
```

```
H=rref([[A b1];eye(6)]')', Q=H(5:10, : ), B=[A b1]*Q
```

```
H =
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
         0         0    1.0000         0         0         0
    2.0000    7.0000   -2.0000         0         0         0
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
    0.5000   -0.5000    0.5000   -1.5000    1.5000   -1.5000
   -1.0000   -2.0000    1.0000    1.0000    1.0000    2.0000
    0.5000    1.5000   -0.5000   -0.5000   -0.5000   -0.5000
Q =
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
    0.5000   -0.5000    0.5000   -1.5000    1.5000   -1.5000
   -1.0000   -2.0000    1.0000    1.0000    1.0000    2.0000
    0.5000    1.5000   -0.5000   -0.5000   -0.5000   -0.5000
B =
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     2     7    -2     0     0     0
```

We use the fourth column of Q to find a solution $\vec{x}$ to $A\,\vec{x} = \vec{b}1$, and verify the result by computing $A\,\vec{x}$.

```
x=Q(1:5, 4)/(-Q(6, 4))
```

```
x =
     2
     0
     0
    -3
     2
```

```
verification=A*x, b1
```

```
verification =
    -1
     2
     3
     6
b1 =
    -1
     2
     3
     6
```

We repeat the computation with the vector $\vec{b}2$.

```
b2=[-1 2 3 5]'

b2 =
    -1
     2
     3
     5

H=rref([[A b2];eye(6)]')', Q=H(5:10, : ), B=[A b2]*Q

H =
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
    -3   -11     3     2    -1    -1
     5    16    -4    -3     0     3
    -7   -24     7     4    -1    -1
     2     7    -2    -1     0     0
Q =
     0     0     0     0     1     0
     0     0     0     0     0     1
    -3   -11     3     2    -1    -1
     5    16    -4    -3     0     3
    -7   -24     7     4    -1    -1
     2     7    -2    -1     0     0
B =
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
```

Observe that none of the columns of Q which generate the zero columns of B, has a non-zero last element. This indicates that $\vec{b}2$ is not in the column space of A and the system $A\,\vec{x} = \vec{b}$ has therefore no solution. This can easily be verified using the partic command or by examining the reduced row echelon form of $[A\,\vec{b}2]$.

```
partic(A, b2)

ans =
     []

rref([A b2])

ans =
    1.0000         0         0    0.3333    1.0000         0
         0    1.0000         0   -0.3333         0         0
         0         0    1.0000         0   -1.0000         0
         0         0         0         0         0    1.0000
```