Math 323
Linear Algebra and Matrix Theory I
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Key Homework 20

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A=[.8 .3; .2 .7], A_s=A^2, A_i=[.6 .6; .4 .4]

```
A =
    0.8000    0.3000
    0.2000    0.7000
A_s =
    0.7000    0.4500
    0.3000    0.5500
A_i =
    0.6000    0.6000
    0.4000    0.4000
```

[V, D]=eig(A), [V_s, D_s]=eig(A_s), [V_I, D_I]=eig(A_i)

```
V =
    0.8321   -0.7071
    0.5547    0.7071
D =
    1.0000         0
         0    0.5000
V_s =
    0.8321   -0.7071
    0.5547    0.7071
D_s =
    1.0000         0
         0    0.2500
V_I =
    0.8321   -0.7071
    0.5547    0.7071
D_I =
     1     0
     0     0
```

Observe that all three matrices have the same linearly independent eigenvectors: $\vec{v}_1 = (0.8321, 0.5547), \vec{v}_2 = (-0.7071, 0.7071)$. The eigenvalues of A^2 are the averages of the corresponding eigenvalues of A and A^∞. This means that for every vector $\vec{x} \in R^2$,

$$A^2\vec{x} = A^2(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) = \alpha_1\vec{v}_1 + 0.25\,\alpha_2\vec{v}_2 = \frac{1}{2}(A + A^\infty)(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) = \frac{1}{2}(A + A^\infty)\vec{x},$$

which implies that $A^2 = \frac{1}{2}(A + A^\infty)$.

a)

```
eig(A), eig(A([2 1], : ))

ans =
    1.0000
    0.5000
ans =
   -0.5000
    1.0000
```

b) A zero eigenvalue indicates that the matrix is singular. Since rank(A) = rank(U) = rank(rref(A)), singularity is not affected by elimination. When A has an eigenvalue zero, the so do U and rref(A).

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```
A=[1 4; 2 3], [V1, D1]=eig(A), [V2, D2]=eig(A+eye(2))

A =
     1     4
     2     3
V1 =
   -0.8944   -0.7071
    0.4472   -0.7071
D1 =
    -1     0
     0     5
V2 =
   -0.8944   -0.7071
    0.4472   -0.7071
D2 =
     0     0
     0     6
```

A+I has the same eigenvectors as A. Its eigenvalues are increased by 1.
Proof: Let λ be an eigenvalue of A associated with the eigenvector $\vec{v}$, then $(A+I)\vec{v} = \lambda\vec{v} + \vec{v} = (\lambda+1)\vec{v}$.

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```
A=[-1 3; 2 0],  [V, D]=eig(A), [V_s, D_s]=eig(A^2)

A =
    -1     3
     2     0
V =
   -0.8321   -0.7071
    0.5547   -0.7071
D =
    -3     0
     0     2
V_s =
    0.8321    0.7071
   -0.5547    0.7071
D_s =
     9     0
     0     4
```

A^2 has the same eigenvectors as A, when A has eigenvalues λ_1, λ_2 the A^2 has eigenvalues $\lambda_1{}^2, \lambda_2{}^2$. Proof: Let A have the eigenvalue λ corresponding to the eigenvector $\vec{v}$, then $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$.

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```
A=[1 0; 1 1]; B=A'; eig(A), eig(B), eigenvalues_product=eig(A).*eig(B),
eigenvalues_AB=eig(A*B), eigenvalues_BA=eig(B*A)
```

```
ans =
     1
     1
ans =
     1
     1
eigenvalues_product =
     1
     1
eigenvalues_AB =
    0.3820
    2.6180
eigenvalues_BA =
    2.6180
    0.3820
```

a) Clearly the eigenvalues of a product of matrices do not equal the product of the eigenvalues of those matrices
b) The eigenvalues of AB however do equal the eigenvalues of BA. Proof: Let $\lambda \neq 0$ denote an eigenvalue of AB corresponding to the eigenvector $\vec{x}$, then $AB\vec{x} = \lambda\vec{x}$ implies that $B\vec{x} \neq \vec{0}$. Moreover $AB\vec{x} = \lambda\vec{x} \Rightarrow BA(B\vec{x}) = B(AB\vec{x}) = B(\lambda\vec{x}) = \lambda(B\vec{x})$, which means that λ is also an eigenvalue of BA (with eigenvector $B\vec{x}$). To complete the argument we observe that if $\lambda = 0$ is an eigenvalue of AB, then AB is singular and also BA is singular, since 0 = det(AB) = det(BA). The latter implies that $\lambda = 0$ is also an eigenvalue of BA

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a) …, the way to find λ is to compute $A\vec{x}$.
b) …, the way to find $\vec{x}$ is to solve $(A - \lambda I)\vec{x} = \vec{0}$.

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a) See the key to problem 4 for a proof.
b) If λ is an eigenvalue of an invertible matrix A then $\lambda \neq 0$ (otherwise the nullspace of A would contain a non-zero element). Moreover $A\vec{x} = \lambda\vec{x} \Rightarrow \vec{x} = A^{-1}A\vec{x} = A^{-1}(\lambda\vec{x}) = \lambda A^{-1}\vec{x} \Rightarrow A^{-1}\vec{x} = \lambda^{-1}\vec{x}$, so λ^{-1} is an eigenvalue of A^{-1}.
c) See the key to problem 2 for a proof.

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```
P=[0.2 0.4 0; 0.4 0.8 0; 0 0 1], [V, D]=eig(P)

P =
    0.2000    0.4000         0
    0.4000    0.8000         0
         0         0    1.0000
V =
    0.8944    0.4472         0
   -0.4472    0.8944         0
         0         0    1.0000
D =
     0     0     0
     0     1     0
     0     0     1
```

The eigenvalues of P^{100} are $1^{100} = 1, \text{ and } 0^{100} = 0$, the eigenvectors are the same as those of P. Observe that P has more than one eigenvector associated with the eigenvalue $\lambda = 1$: $\vec{v}_1 = (0.4472,\ 0.8944,\ 0)$ (you could say $\vec{v}_1 = (1,\ 2,\ 0)$) and $\vec{v}_2 = (0,0,1)$. This means that any linear combination of $\vec{v}_1$ and $\vec{v}_2$ is an eigenvector associated with that eigenvalue. In particular $\vec{v}_1 + \vec{v}_2$ is such eigenvector with no zero components.

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```
u=[1/6 1/6 3/6 5/6]', P=u*u'

u =
    0.1667
    0.1667
    0.5000
    0.8333
P =
    0.0278    0.0278    0.0833    0.1389
    0.0278    0.0278    0.0833    0.1389
    0.0833    0.0833    0.2500    0.4167
    0.1389    0.1389    0.4167    0.6944
```

a) Since u has length 1, $\vec{u}^T\vec{u} = 1$ and $P\vec{u} = \vec{u}\vec{u}^T\vec{u} = \vec{u}1 = \vec{u}$.

b) If $\vec{u}^T\vec{v} = 0$, then $P\vec{v} = \vec{u}\,\vec{u}^T\vec{v} = \vec{u}0 = \vec{0}$.

c)

```
nulbasis(P)

ans =
    -1    -3    -5
     1     0     0
     0     1     0
     0     0     1
```

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Observe that if A denotes an n by n matrix, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n with leading coefficient $(-1)^n$. (just look at the leading term of

$p(\lambda) = \det(A - \lambda I) = \sum \det(P)\,(A-\lambda I)_{1,j_1} \cdots (A-\lambda I)_{n,j_n}\,)$. This implies that $p(\lambda) = \det(A - \lambda I) = \prod_{j=1}^{n} (\lambda_j - \lambda)$. Now let $\lambda = 0$ to obtain $\det(A) = \prod_{j=1}^{n} \lambda_j$.

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a) $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$

b) .. and $\lambda = (a + d - \sqrt{\cdots})/2$

c) … their sum is $a + d$.

This result will be generalized in the section 6.2.

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a) Eigenvectors associated with distinct eigenvalues are linearly independent, so the sum of the dimensions of the eigenspaces of any n by n matrix cannot be greater than n. Since the 3 by 3 matrix B has three distinct eigenvalues, the dimension of each of the eigenspaces has to be exactly one. Since the eigenspace associated with the eigenvalue $\lambda = 0$ is exactly the nullspace of B, this means that the rank of B equals $3 - 1 = 2$.

b) Since rank(B) = 2 and B is 3 by 3, we know that det(B) = 0 an therefore $\det(B^T B) = \det(B^T)\det(B) = 0$.

c) The eigenvalues of $B^T B$ are not uniquely determined by the eigenvalues of B. As an example of this fact consider the matrices $B_1 = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}$ and $B_2 = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{matrix}$. Both matrices have eigenvalues 0, 1, and 2, but the eigenvalues of $B_1^T B_1$ are 0, 1, and 4, while the eigenvalues of $B_2^T B_2$ are $0, 3 + \sqrt{5}$, and $3 - \sqrt{5}$.

d) The eigenvalues of $(B + I)^{-1}$ are the reciprocals of the eigenvalues of $B + I$ (which are 1, 2, and 3). The desired eigenvalues are therefore, $1, \frac{1}{2}$ and $\frac{1}{3}$.

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Because all eigenvalues are zero, both the trace and the determinant of the matrix need to be zero. Observe that in the 2 by 2 case, these equations are not only a necessary, but also a sufficient condition for zero to be the only eigenvalue. If $A = \begin{matrix} a & b \\ c & -a \end{matrix}$, the equation det(A) = 0 quickly leads to the solutions {a = a, c = c, b = $-\frac{a^2}{c}$}, or {a = 0, c = 0, b = b}. It is a matter of elementary arithmetic to show that in each of these two cases $A^2 = O$. Examples are $\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$, $\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$, *and* $\begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}$ for {a, b, c} equal to {0, 1, 0}, {0, 0, 1}, and {1, -1, 1} respectively.