Math 323
Linear Algebra and Matrix Theory I
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Key Homework 21

Strang Page 266 no: 3

```
S=[1 1; 0 1], L=[2 0; 0 5], A=S*L*inv(S)

S =
     1     1
     0     1
L =
     2     0
     0     5
A =
     2     3
     0     5
```

Strang Page 266 no: 5

a) Not necessarily, because A could have an eigenvalue zero.
b) Yes, because A has a full set of linearly independent eigenvectors.
c) Yes, because the columns of S are linearly independent.
d) Not necessarily, but constructing a counter example takes some effort. We start with an invertible, yet not diagonalizable matrix S.

```
S=[2 1 0; -1 0 1; 1 3 1], Sinv=inv(S), [vS, dS]=eig(S)

S =
     2     1     0
    -1     0     1
     1     3     1
Sinv =
    0.7500    0.2500   -0.2500
   -0.5000   -0.5000    0.5000
    0.7500    1.2500   -0.2500
vS =
  -0.1961             0.0000 + 0.7071i   0.0000 - 0.7071i
   0.5883            -0.0000 + 0.0000i  -0.0000 - 0.0000i
  -0.7845             0.0000 + 0.7071i   0.0000 - 0.7071i
dS =
  -1.0000                  0                  0
        0             2.0000 + 0.0000i        0
        0                  0             2.0000 - 0.0000i
```

Now we create a 3 by 3 matrix A such that the eigenvectors of A are the columns of S, with eigenvalues 1, 2 and 3 respectively. (The 1, 2 and 3 are arbitrarily chosen integers). To do this, let B denote the matrix whose columns are the columns of S multiplied by 1, 2, and 3 respectively, then AS = B and $A = BS^{-1}$.

```
B=[S(:, 1) 2*S(:, 2) 3*S(:,3 )], A=B*Sinv

B =
     2     2     0
    -1     0     3
     1     6     3
A =
    0.5000   -0.5000    0.5000
    1.5000    3.5000   -0.5000
         0    1.0000    2.0000
```

This matrix A has the property that the matrix S with the eigenvectors of A as its columns, is invertible, yet not diagonalizable.

Strang Page 266 no: 6

… A is a diagonal matrix.
… It followes directly from the Gauss Jordan elimnation process, that if S is (upper or lower) triangular and invertible, then S^{-1} is also (upper or lower) triangular. In this case so are SΛ and SΛS^{-1}.

Strang Page 266 no: 8

Similar to the procedure followed in problem 5, let $S = \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}$ and $B = \begin{matrix} a & b \\ a & -b \end{matrix}$, then

$$A = BS^{-1} = \begin{matrix} \frac{a}{2}+\frac{b}{2} & \frac{a}{2}-\frac{b}{2} \\ \frac{a}{2}-\frac{b}{2} & \frac{a}{2}+\frac{b}{2} \end{matrix} = \begin{matrix} c & d \\ d & c \end{matrix}.$$

Strang Page 266 no: 12

Look at the characteristic polynomial of the Fibonacci matrix $A = \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$, $p(\lambda) = \lambda^2 - \lambda - 1$.

This implies that $\lambda_i^2 = \lambda_i + 1 \Rightarrow \lambda_i^{k+2} = \lambda_i^{k+1} + \lambda_i^k, i = 1,2, k = 1,2,\cdots$.

Strang Page 266 no: 13

… $A^{20}(\vec{x}_1 + \vec{x}_2) = \lambda_1^{20}\vec{x}_1 + \lambda_2^{20}\vec{x}_2$ and its second component equals $\lambda_1^{20} + \lambda_2^{20}$, which is easily computed to be:

```
format long
FN_20=((1+sqrt(5))/2)^20+((1-sqrt(5))/2)^20

FN_20 =
    1.512700000000001e+004
```

Strang Page 266 no: 15

a) True! Because the matrix A does not have zero as an eigenvalue.
b) False! Because, the matrix could either be, or not be, diagonalizable. In the example below the matrix A_1 is diagonalizable, but the matrix A_2 is not. However both matrices have 2, 2, and 5 as their eigenvectors.

$$A_1 = \begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{matrix} \quad \text{and} \quad A_2 = \begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{matrix}$$

We check this claim using MATLAB.

```
A1=[2 0 0; 0 2 0; 0 0 5], [V1, D1]=eig(A1)

A1 =
     2     0     0
     0     2     0
     0     0     5
V1 =
     1     0     0
     0     1     0
     0     0     1
D1 =
     2     0     0
     0     2     0
     0     0     5
```

While

```
A2=[2 1 0; 0 2 0; 0 0 5], [V2, D2]=eig(A2)

A2 =
     2     1     0
     0     2     0
     0     0     5
V2 =
    1.0000   -1.0000         0
         0    0.0000         0
         0         0    1.0000
D2 =
     2     0     0
     0     2     0
     0     0     5
```

c) False! See the answer to (b).

Strang Page 266 no: 18

The matrix $A = \begin{matrix} 3 & 1 \\ 0 & 3 \end{matrix}$, is not diagonalizable because the rank of $(A - 3I)$ equals one (and not zero).

```
A=[3 1; 0 3], r=rank(A-3*eye(2))

A =
     3     1
     0     3
r =
     1
```

Changing one of the 3's will create distinct eigenvalues, and linearly independent eigenvectors. Changing the 0 will have the same effect.

Observe that changing the 1 will have NO effect on $\det(A - \lambda I)$, so the one and only eigenvalue will remain 3. Moreover, if the change is just 0.1, the rank of $(A - 3I)$ will still be one.

However if we were to change the one to a zero, then the rank of $(A - 3I)$ would be zero and the matrix would be diagonalizable. (in fact: it would already be diagonal)

Strang Page 266 no: 26

… since the left side has trace zero. (the diagonal elements of AB are equal to the diagonal elements of BA)

We find a scalar a such that if $E = \begin{matrix} 1 & 0 \\ a & 1 \end{matrix}$, then $EE^T - E^T E = \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}$. This equation implies that $\begin{matrix} -a^2 & 0 \\ 0 & a^2 \end{matrix} = \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}$. We can take a equal to 1 or -1.

Strang Page 266 no: 31

```
A=[5 4; 4 5], [VA, DA]=eig(A)

A =
     5     4
     4     5
VA =
    0.7071    0.7071
   -0.7071    0.7071
DA =
     1     0
     0     9
```

We now compute a matrix square root of A and check the result.

```
RA=VA*sqrt(DA)*inv(VA), checkA=RA^2

RA =
     2     1
     1     2
checkA =
     5     4
     4     5
```

Of course the matrix B has no real matrix square root, because the square root of -1 is not real. However a complex square root can easily be computed.

```
B=[4 5; 5 4], [VB, DB]=eig(B), RB=VB*sqrt(DB)*inv(VB), checkB=RB^2

B =
     4     5
     5     4
VB =
    0.7071    0.7071
   -0.7071    0.7071
DB =
    -1     0
     0     9
RB =
   1.5000 + 0.5000i   1.5000 - 0.5000i
   1.5000 - 0.5000i   1.5000 + 0.5000i
checkB =
     4     5
     5     4
```

ATLAST Page 142 no: 1

Unfortunately these matrices are not in the list of our **eigshow** command. (Sorry !!!) All we can do is: compute the eigenvalues and eigenvectors and use the so generated information to answer the questions.

```
A=[2 0; 1.5 0.5], [V, D]=eig(A)

A =
    2.0000         0
    1.5000    0.5000
V =
         0    0.7071
    1.0000    0.7071
D =
    0.5000         0
         0    2.0000
```

1a) The eigenvalues are distinct, so the eigenvectors are linearly independent, and the eigenvalues are real and non-zero, so $\vec{x}$ and $A\vec{x}$ will line up four times
1b) The eigenvalues are 0.5 and 2.
1d) The eigenvectors are linearly independent, so A is diagonalizable.

```
A=[-2 1; 1 -2], [V, D]=eig(A)

A =
    -2     1
     1    -2
V =
    0.7071    0.7071
   -0.7071    0.7071
D =
   -3.0000         0
         0   -1.0000
```

2a) The eigenvalues are again distinct, so the eigenvectors are linearly independent, and the eigenvalues are real and non-zero , so $\vec{x}$ and $A\vec{x}$ will line up four times.
2b) The eigenvalues are -3 and -1.
2d) The eigenvectors are linearly independent, so A is diagonalizable.

```
A=[-1 2; -2 4], [V, D]=eig(A)

A =
    -1     2
    -2     4
V =
   -0.8944   -0.4472
   -0.4472   -0.8944
D =
     0     0
     0     3
```

3a) The eigenvalues are again distinct, so the eigenvectors are linearly independent, moreover since the eigenvalues are real, but one of them is zero, $\vec{x}$ and $A\vec{x}$ will line up two times.
3b) The eigenvalues are 0 and 3.
3d) The eigenvectors are linearly independent, so A is diagonalizable.

```
A=[1 -1; 1 1], [V, D]=eig(A)

A =
     1    -1
     1     1
V =
  -0.7071                 -0.7071
        0 + 0.7071i             0 - 0.7071i
D =
   1.0000 + 1.0000i             0
        0                  1.0000 - 1.0000i
```

4a) The eigenvalues are distinct, so the eigenvectors are linearly independent, however the eigenvalues are complex, so $\vec{x}$ and $A\vec{x}$ will never line up.
4b) The eigenvalues are 1 + i and 1 - i.
4d) The eigenvectors are linearly independent, so A is diagonalizable.

ATLAST Page 142 no: 5

a)

```
A=gallery(5)

A =
          -9          11         -21          63        -252
          70         -69         141        -421        1684
        -575         575       -1149        3451      -13801
        3891       -3891        7782      -23345       93365
        1024       -1024        2048       -6144       24572

e=eig(A)

e =
  -0.0328 + 0.0243i
  -0.0328 - 0.0243i
   0.0130 + 0.0379i
   0.0130 - 0.0379i
   0.0396

A5=A^5, e5=eig(A5)

A5 =
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
e5 =
     0
     0
     0
     0
     0
```

Obviously this contradicts our expectation that that the eigenvalues of A^5 are the fifth power of the eigenvalues of A. (the discrepancy occurs because of difficulties with the numeric calculation of the eigenvalues of A)

b)

```
p=poly(A), e=roots(p)

p =
    1.0000   -0.0000    0.0000   -0.0000    0.0000   -0.0000
e =
  -0.0328 + 0.0243i
  -0.0328 - 0.0243i
   0.0130 + 0.0379i
   0.0130 - 0.0379i
   0.0396
```

These results are equally unsatisfactory.

c) Since the matrix A has integer entries, the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ must have integer coefficients.

```
p=round(p), roots(p)

p =
     1     0     0     0     0     0
ans =
     0
     0
     0
     0
     0
```

That's more like it !!!

The roots of a polynomial can be extremely sensitive to changes in the coefficients. A very small change (roundoff error) in the coefficients may result in a significant change in the roots. That is what causes the erroneous answers in (a) and (b).

d) Proof: Let B be nilpotent of order k and suppose that B has an non-zero eigenvalue λ. Then $\exists\ \vec{x} \neq \vec{0}$ such that $B\vec{x} = \lambda\vec{x} \Rightarrow B^k\vec{x} = \lambda^k\vec{x} \neq \vec{0}$. This clearly contradicts the fact that B is nilpotent of order k. Therefore our assumption that B has a non-zero eigenvalue must be false, and zero is the only eigenvalue of B.

e)

```
R=rref(A)

R =
    1.0000         0         0         0         0
         0    1.0000         0         0   -0.0820
         0         0    1.0000         0    0.5547
         0         0         0    1.0000   -3.8008
         0         0         0         0         0
```

Clearly rank(A) = 4, and the nullity (that is the dimension of the nullspace) of A is 5 - 4 = 1. This implies that the geometric multiplicity of the eigenvalue $\lambda = 0$ equals 1, which is nowhere near its algebraic multiplicity 5, so A is not diagonalizable.