Math 323
Linear Algebra and Matrix Theory I
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Key Homework 8

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```
A=[1 0; 2 1], B=[1 3; 0 1], C=[1 3; 2 7], ABT=(A*B)', BTAT=B'*A',
ATBT=A'*B'

A =
     1     0
     2     1
B =
     1     3
     0     1
C =
     1     3
     2     7
ABT =
     1     2
     3     7
BTAT =
     1     2
     3     7
ATBT =
     7     2
     3     1
```

Suppose that $AB = BA$ then $(AB)^T = (BA)^T \Rightarrow B^T A^T = A^T B^T$.

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```
A=[0 0; 1 0], A^2

A =
     0     0
     1     0
ans =
     0     0
     0     0
```

This clearly illustrates that $A^2 = 0$ is possible even for a non zero matrix A.

On the other hand $A^T A = 0$ implies that all entries of the $A^T A$ matrix are zero, including the elements on the main diagonal. Those zero diagonal elements are the dot products of the columns of A with themselves, so they are the squares of the lengths of the columns of A. This means that the columns of A are all zero and that A is truly the zero matrix.

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a)
We use a matrix which is symmetric but it is not invertible because it does not have a full set of pivots.

```
A=[1 1; 1 1]

A =
     1     1
     1     1
```

b)
Use a symmetric matrix which is invertible, while LU decomposition fails because a row exchange is required in the Gaussian elimination process.

```
A=[0 1; 1 1]

A =
     0     1
     1     1
```

A PLU decomposition so that PA = LU is of course possible as demonstrated below.

```
[P, L, U]=splu(A), PA=P*A, LU=L*U

Pivots in rows:
     2     1
P =
     0     1
     1     0
L =
     1     0
     0     1
U =
     1     1
     0     1
PA =
     1     1
     0     1
LU =
     1     1
     0     1
```

c)
Use a symmetric matrix for which an LU decomposition is possible, but LL^T decomposition fails because the diagonal elements of U are not all ones.

```
A=[1 1; 1 0]
```

```
A =
     1     1
     1     0
```

We demonstrate the LU decomposition.

```
[L, U]=slu(A)
```

```
L =
     1     0
     1     1
U =
     1     1
     0    -1
```

Of course an LDL^T decomposition is still possible as demonstrated next.

```
L, D=diag(diag(U)), U=inv(D)*U, A, LDU=L*D*U
```

```
L =
     1     0
     1     1
D =
     1     0
     0     1
U =
     1     1
     0     1
A =
     1     1
     1     0
LDU =
     1     1
     1     2
```

Observe that the new U is truly L^T so the LDL^T decomposition has been achieved.

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a)

```
A=[1 3; 3 2], [L, U]=slu(A); D=diag(diag(U)); LT=L'; L, D, LT
```

```
A =
      1             3
      3             2
L =
      1             0
      3             1
D =
      1             0
      0            -7
LT =
      1             3
      0             1
```

Observe again that $U = L^T$.

b)
Since the matrix contains variables, we perform Gaussian elimination manually.

$$\begin{matrix} 1 & b \\ b & c \end{matrix} \Rightarrow \begin{matrix} 1 & b \\ 0 & c-b^2 \end{matrix} \Rightarrow L = \begin{matrix} 1 & 0 \\ b & 1 \end{matrix}, \quad D = \begin{matrix} 1 & 0 \\ 0 & c-b^2 \end{matrix}, \quad and \quad U = D^{-1} \quad times \quad \begin{matrix} 1 & b \\ 0 & c-b^2 \end{matrix}$$

$\Rightarrow U = \begin{matrix} 1 & b \\ 0 & 1 \end{matrix}$ which of course equals L^T.

c)

```
format rat
A=[2 -1 0; -1 2 -1; 0 -1 2], [L, U]=slu(A); D=diag(diag(U)); LT=L'; L,
D, LT
```

```
A =
       2            -1             0
      -1             2            -1
       0            -1             2
L =
       1             0             0
    -1/2             1             0
       0          -2/3             1
D =
       2             0             0
       0           3/2             0
       0             0           4/3
LT =
       1          -1/2             0
       0             1          -2/3
       0             0             1
```

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a) If B is a square matrix , then

$$(B+B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$$

This implies that the matrix $B+B^T$ is symmetric.

b) If B is a square matrix , then

$$(B-B^T)^T = B^T - (B^T)^T = B^T - B = -(B-B^T)$$

This implies that the matrix $B-B^T$ is skew - symmetric.

c)

```
B=randint(4), A=B+B'
```

```
B =
     9     7     6     8
    -5     5    -1     5
     2    -1     2    -6
     0    -9     6    -2
A =
    18     2     8     8
     2    10    -2    -4
     8    -2     4     0
     8    -4     0    -4
```

Observe that the i-th row of A equals the i-th column of A.

d)

```
B=randint(4), A=B-B'
```

```
B =
     8    -8    -7    -4
     8    -3    -6    -6
    -2     6    -6    -9
     7    -9     2     5
A =
     0   -16    -5   -11
    16     0   -12     3
     5    12     0   -11
    11    -3    11     0
```

Observe that the i-th row of A equals negative the i-th column of A. The diagonal elements of A are of course zero because $a_{ii} = -a_{ii}$ can be satisfied only if $a_{ii} = 0$.

e)

- i. True, since if A and B are symmetric: $(A+B)^T = A^T + B^T = A + B$
- ii. True, since if A is symmetric: $(cA)^T = cA^T = cA$
- iii. Not true. An example follows.

```
B=randint(2); A1=B+B',  B=randint(2); A2=B+B',  A1A2=A1*A2
```

```
A1 =
    10     2
     2     2
A2 =
    12    -6
    -6   -18
A1A2 =
   108   -96
    12   -48
```

- iv. True, because if A is symmetric: $A^T = A$.
- v. True, because if A is symmetric:$(A^{-1})^T = (A^T)^{-1} = A^{-1}$

f)

- i. True, since if A and B are skew-symmetric:
  $(A+B)^T = A^T + B^T = -A - B = -(A+B)$
- ii. True, since if A is skew symmetric: $(cA)^T = cA^T = c(-A) = -cA$

- iii. Not true. An example follows.

```
B=randint(2); A1=B-B', B=randint(2); A2=B-B', A1A2=A1*A2

A1 =
      0             3
     -3             0
A2 =
      0            11
    -11             0
A1A2 =
    -33             0
      0           -33
```

- iv. True, because if A is skew-symmetric then $A^T = -A \Rightarrow (A^T)^T = (-A)^T = -A^T$.

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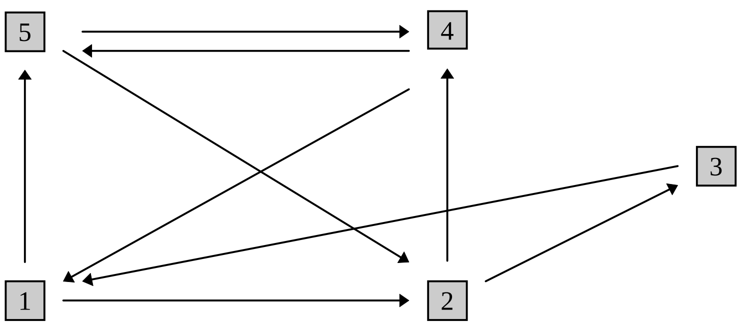
a)



b)

```
format short
A=[0 1 0 0 1; 0 0 1 1 0; 1 0 0 0 0; 1 0 0 0 1; 0 1 0 1 0]

A =
     0     1     0     0     1
     0     0     1     1     0
     1     0     0     0     0
     1     0     0     0     1
     0     1     0     1     0
```

c)

```
A_square=A^2

A_square =
     0     1     1     2     0
     2     0     0     0     1
     0     1     0     0     1
     0     2     0     1     1
     1     0     1     1     1
```

Observe that the second row of A lists all the flights which depart from San Francisco, while the first column of A lists all the flights which arrive in Los Angeles. The 2 in the (2, 1) of A^2 indicates the two ways a passenger can travel from San Francisco to Los Angeles. The passenger can be routed through Fresno, or though Monterey.

d)

```
A+A^2

ans =
     0     2     1     2     1
     2     0     1     1     1
     1     1     0     0     1
     1     2     0     1     2
     1     1     1     2     1
```

The (i, j) element of this matrix indicates in how many ways one can travel from i to j with atmost one stop on the way.

e)

Clearly two flight segments will not be enough for instance to go from Montery to Fresno, since $(A+A^2)_{34}=0$. We now investigate $A+A^2+A^3$.

```
A+A^2+A^3

ans =
     3     2     2     3     3
     2     3     1     2     3
     1     2     1     2     1
     2     3     2     4     3
     3     3     1     3     3
```

Since this matrix contains no zeros (on off diagonal elements) it is possible to get from any city to any other city served by this airline, by using at most three flight segments.

f)

```
B=A; B(5, 2)=0, less_than_three_flight_segments=B+B^2+B^3,
less_than_four_flight_segments=B+B^2+B^3+B^4

B =
     0     1     0     0     1
     0     0     1     1     0
     1     0     0     0     0
     1     0     0     0     1
     0     0     0     1     0
less_than_three_flight_segments =
     3     1     1     2     3
     2     2     1     2     3
     1     1     1     2     1
     2     1     1     3     3
     1     1     0     2     2
```

```
less_than_four_flight_segments =
     3     4     1     4     6
     3     2     3     6     4
     4     1     1     2     3
     5     2     1     4     6
     2     1     1     4     3
```

Clearly if the flight form Sacramento to San Francisco is removed, it will take at least four flight segments to travel from Sacramento to Monterey.

g)
No, it is not possible to add just one non stop flight to the schedule and be able to fly between Monterey and Fresno using at most two flight segments. Because to enable such a connection from Montery to Fresno we need to either change the (3, 4) element of A or the (3, 4) element of A^2. This implies that a 1 needs to be added to either the third row or the fourth column of A. To enable a similar connection from Fresno to Monterey we need to add a 1 to either the fourth row or the third column of A. This cannot be done by adding just one 1 to the matrix A unless we change the (3, 3) or (4, 4) elements of A to 1, which is of course makes to sense.

The simplest solution is to add non-stop flights between Fresno and Monterey. The computation below shows the result of that addition.

```
A(3, 4)=1; A(4, 3)=1, less_than_two_flight_segments=A+A^2

A =
     0     1     0     0     1
     0     0     1     1     0
     1     0     0     1     0
     1     0     1     0     1
     0     1     0     1     0
less_than_two_flight_segments =
     0     2     1     2     1
     2     0     2     2     1
     2     1     1     1     2
     2     2     1     2     2
     1     1     2     2     1
```