Math 323
Linear Algebra and Matrix Theory I
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Lesson 5
LU and LDU Decomposition
Forward and Backward Substitution

5.1 Matrix Operations

- Example 5.1.1
  If A = [2 3 -5; 1 2 7; 3 -9 4] and **b**=[5; 2; 7] , use the **slu** command to factor the matrix A as the product of a lower triangular matrix L with ones on the main diagonal and an upper triangular matrix U. Interpret the result and use the LU factorization to solve the system A**x** = **b**. Finally factor the matrix A as the product of a lower triangular matrix L with ones on the main diagonal, a diagonal matrix D and an upper triangular matrix U with ones on the main diagonal.

First we enter the data, then use the **slu** command to compute the factorization.

```
format rat

A=[2 3 -5; 1 2 7; 3 -9 4], b=[5; 2; 7]

A =
      2            3            -5
      1            2             7
      3           -9             4
b =
      5
      2
      7

[L, U]=slu(A)

L =
      1            0             0
     1/2           1             0
     3/2         -27             1
```

```
U =
      2             3            -5
      0            1/2          19/2
      0             0           268
```

We check the factorization.

```
A, LU=L*U
```

```
A =
      2             3            -5
      1             2             7
      3            -9             4
LU =
      2             3            -5
      1             2             7
      3            -9             4
```

In class we will prove that the matrix U is the result of applying Gaussian elimination to the matrix A, while the below diagonal part of L contains the multipliers involved in the Gaussian elimination process. The **slu** command will succeed, provided that **no row exchanges** are required to complete Gaussian elimination. The case that row exchanges are needed is treated in the next lesson.

We now proceed with the solution of A**x** = **b** using successive forward and backward substitution on the equation LU**x** = **b**. We use **MATLAB**'s \ (left division) operator to first solve L**c** = **b** and then proceed with U**x** = **c**.

```
c=L\b, x=U\c
```

```
c =
      5
    -1/2
    -14
x =
   319/134
    -1/134
    -7/134
```

Finally we extract the diagonal of the matrix U and place the data on the diagonal of the diagonal matrix D. The tool we use is **MATLAB**'s **diag** command. Applied to a square matrix, the diag command produces a column vector containing the diagonal elements of that matrix. Applied to a row or a column vector, it produces a diagonal matrix with the specified entries on the main diagonal.

```
U, D=diag(diag(U))
```

```
U =
      2             3            -5
      0            1/2          19/2
      0             0           268
D =
      2             0             0
      0            1/2            0
      0             0           268
```

Now we apply the simple observation A = LU = L D (inv(D)) U = L D (inv(D)U) and rename the product inv(D)U as a new matrix U with ones on the main diagonal.

```
L, D, U=inv(D)*U

L =
      1            0            0
     1/2           1            0
     3/2          -27           1
D =
      2            0            0
      0           1/2           0
      0            0          268
U =
      1           3/2         -5/2
      0            1           19
      0            0            1
```

Finally we check the result and compare A to the result of the product LDU.

```
A, LDU=L*D*U

A =
      2            3           -5
      1            2            7
      3           -9            4
LDU =
      2            3           -5
      1            2            7
      3           -9            4
```

- Example 5.1.2

The major advantage of storing an LU decomposition as opposed to storing an inverse matrix is the fact that with LU decomposition, the structure of a banded matrix is preserved, while the inverse of a banded matrix is usually completely filled with non zero elements. This observation is enormously important in practical applications, which often involve banded systems with thousands of equations and thousands of variables.

The result follows immediately from the make up of the L and U matrices. The below diagonal elements of L are the multipliers in the corresponding Gaussian elimination process, therefore if the coefficient matrix A has a row which starts with a string of zeros below the main diagonal, those zeros are preserved in the matrix L.

Similarly, the matrix U is obtained from the coefficient matrix A by Gaussian elimination, therefore if the matrix A has a column which starts with a string of zeros above the main diagonal, those zeros are preserved in the matrix U.

As an example we consider the matrix A=[2 1 0 0 0; 1 2 1 0 0; 0 1 2 1 0; 0 0 1 2 1; 0 0 0 1 2]. Compute the LU decomposition of A and then look at the inverse of A.

First we enter the data, then use the **slu** command to compute the factorization.

```
A=[2 1 0 0 0; 1 2 1 0 0; 0 1 2 1 0; 0 0 1 2 1; 0 0 0 1 2], [L, U]=slu(A)
```

```
A =
      2          1          0          0          0
      1          2          1          0          0
      0          1          2          1          0
      0          0          1          2          1
      0          0          0          1          2
L =
      1          0          0          0          0
     1/2         1          0          0          0
      0         2/3         1          0          0
      0          0         3/4         1          0
      0          0          0         4/5         1
U =
      2          1          0          0          0
      0         3/2         1          0          0
      0          0         4/3         1          0
      0          0          0         5/4         1
      0          0          0          0         6/5
```

Observe that the banded structure is nicely preserved. On the other hand the inverse of the matrix A is completely filled with non zero elements.

```
invA=inv(A)
```

```
invA =
     5/6       -2/3        1/2       -1/3        1/6
    -2/3        4/3       -1          2/3       -1/3
     1/2       -1          3/2       -1          1/2
    -1/3        2/3       -1          4/3       -2/3
     1/6       -1/3        1/2       -2/3        5/6
```