Key Homework 16

Math 277, Fall 2005

Ordinary Differential Equations

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Initializations

```
> restart;
```

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Code the differential equation and compute the auxiliary equation.

```
> deq:=diff(y(t),t$2)+2*diff(y(t), t)+5*y(t)=0;
```

$$deq := \left(\frac{d^2}{dt^2}\,y(t)\right) + 2\left(\frac{d}{dt}\,y(t)\right) + 5\,y(t) = 0$$

```
> aux_eq:=simplify(eval(subs(y(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := r^2 + 2\,r + 5 = 0$$

Solve the auxiliary equation and find the eigenvalues.

```
> evals:=solve(aux_eq, r);
```

$$evals := -1 + 2\,I, -1 - 2\,I$$

Code the corresponding solutions and create the general solution.

```
> Y[1]:=exp(Re(evals[1])*t)*cos(Im(evals[1])*t);
  Y[2]:=exp(Re(evals[1])*t)*sin(Im(evals[1])*t);
```

$$Y_1 := \mathbf{e}^{(-t)}\cos(2\,t)$$

$$Y_2 := \mathbf{e}^{(-t)}\sin(2\,t)$$

The general solution is given by an arbitrary linear combination of Y_1 and Y_2.

```
> ans[13]:=yg=add(c[k]*Y[k], k=1..2);
```

$$ans_{13} := yg = c_1\,\mathbf{e}^{(-t)}\cos(2\,t) + c_2\,\mathbf{e}^{(-t)}\sin(2\,t)$$

```
>
```

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To solve the given initial value problem, we first compute the general solution of the differential equation and then implement the initial conditions.

The general solution of the differential equation

Code the differential equation and compute the auxiliary equation.

```
> deq:=diff(y(t),t$2)+2*diff(y(t), t)+17*y(t)=0;
```

$$deq := \left(\frac{d^2}{dt^2}\,y(t)\right) + 2\left(\frac{d}{dt}\,y(t)\right) + 17\,y(t) = 0$$

```
> aux_eq:=simplify(eval(subs(y(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := r^2 + 2\,r + 17 = 0$$

Solve the auxiliary equation and find the eigenvalues.

```
> evals:=solve(aux_eq, r);
```

$$evals := -1 + 4\,I, -1 - 4\,I$$

Code the corresponding solutions and create the general solution.

```
> Y[1]:=exp(Re(evals[1])*t)*cos(Im(evals[1])*t);
  Y[2]:=exp(Re(evals[1])*t)*sin(Im(evals[1])*t);
```

$$Y_1 := \mathbf{e}^{(-t)}\cos(4\,t)$$

$$Y_2 := e^{(-t)} \sin(4\,t)$$

The general solution is given by an arbitrary linear combination of Y_1 and Y_2.

```
> yg:=add(c[k]*Y[k], k=1..2);
```

$$yg := c_1\, e^{(-t)} \cos(4\,t) + c_2\, e^{(-t)} \sin(4\,t)$$

```
>
```

Implementation of the initial conditions y(0) = 1, y'(0) = -1

```
> eq1:=eval(subs(t=0, yg))=1;
  eq2:=eval(subs(t=0, diff(yg, t)))=-1;
  val_c:=solve({eq1, eq2}, {c[1], c[2]});
```

$$eq1 := c_1 = 1$$

$$eq2 := -c_1 + 4\,c_2 = -1$$

$$val\_c := \{c_1 = 1, c_2 = 0\}$$

The solution of the initial value problem is given by

```
> ans[22]:=y=subs(val_c, yg);
```

$$ans_{22} := y = e^{(-t)} \cos(4\,t)$$

```
>
```

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The initial value problem

$$y'' + 100\,y' + y = 0,\ y(0) = 1,\ y'(0) = 0$$

represents a damped mass-spring system with a very large damping coefficient $b = 100$. Since $0 < b^2 - 4\,m\,k$, the solution will not oscillate, but it will exponentially approach zero. We verify these observations by solving the initial value problem in a manner similar to that used in Problem 22.

The general solution of the differential equation

Code the differential equation and compute the auxiliary equation.

```
> deq:=diff(y(t),t$2)+100*diff(y(t), t)+y(t)=0;
```

$$deq := \left(\frac{d^2}{dt^2} y(t)\right) + 100 \left(\frac{d}{dt} y(t)\right) + y(t) = 0$$

```
> aux_eq:=simplify(eval(subs(y(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := r^2 + 100\,r + 1 = 0$$

Solve the auxiliary equation and find the eigenvalues.

```
> evals:=solve(aux_eq, r);
```

$$evals := -50 + 7\sqrt{51},\ -50 - 7\sqrt{51}$$

Since the eigenvalues are real and distinct, the corresonding solutions are exponential functions.

```
> for k to 2 do
  Y[k]:=exp(evals[k]*t);
  od;
```

$$Y_1 := e^{((-50 + 7\sqrt{51})\,t)}$$

$$Y_2 := e^{((-50 - 7\sqrt{51})\,t)}$$

The general solution is given by a arbitrary linear combination of Y_1 and Y_2.

```
> yg:=add(c[k]*Y[k], k=1..2);
```

$$yg := c_1\, e^{((-50 + 7\sqrt{51})\,t)} + c_2\, e^{((-50 - 7\sqrt{51})\,t)}$$

```
>
```

Implementation of the initial conditions y(0) = 1, y'(0) = 0

```
> eq1:=eval(subs(t=0, yg))=1;
  eq2:=eval(subs(t=0, diff(yg, t)))=0;
  val_c:=solve({eq1, eq2}, {c[1], c[2]});
```

$$eq1 := c_1 + c_2 = 1$$

$$eq2 := c_1\,(-50 + 7\sqrt{51}\,) + c_2\,(-50 - 7\sqrt{51}\,) = 0$$

$$val\_c := \{c_2 = -\frac{25\sqrt{51}}{357} + \frac{1}{2},\ c_1 = \frac{25\sqrt{51}}{357} + \frac{1}{2}\}$$

The solution of the initial value problem is given by

```
> ans[`31b`]:=y=subs(val_c, yg);
  evalf(ans[`31b`]);
```

$$ans_{31b} := y = \left(\frac{25\sqrt{51}}{357} + \frac{1}{2}\right) \mathbf{e}^{((-50+7\sqrt{51}\,)\,t)} + \left(-\frac{25\sqrt{51}}{357} + \frac{1}{2}\right) \mathbf{e}^{((-50-7\sqrt{51}\,)\,t)}$$

$$y = 1.000100030\ \mathbf{e}^{(-0.01000100\,t)} - 0.0001000300\ \mathbf{e}^{(-99.98999900\,t)}$$

Since, in both terms, the exponents are negative

$$\lim_{t\to\infty} y(t) = 0.$$

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32a

We solve the initial value problem $10\,y'' + 250\,y = 0$, $y(0) = .3$, $y'(0) = -.1$

The general solution of the differential equation

Code the differential equation and compute the auxiliary equation.

```
> deq:=10*diff(y(t),t$2)+250*y(t)=0;
```

$$deq := 10\left(\frac{d^2}{dt^2}\,y(t)\right) + 250\,y(t) = 0$$

```
> aux_eq:=simplify(eval(subs(y(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := 10\,r^2 + 250 = 0$$

Solve the auxiliary equation and find the eigenvalues.

```
> evals:=solve(aux_eq, r);
```

$$evals := 5\,I,\ \text{-}5\,I$$

Since the eigenvalues purely imaginary, the corresonding solutions are trigonometric functions.

```
> Y[1]:=cos(Im(evals[1])*t);
  Y[2]:=sin(Im(evals[1])*t);
```

$$Y_1 := \cos(5\,t)$$

$$Y_2 := \sin(5\,t)$$

The general solution is given by an arbitrary linear combination of Y_1 and Y_2.

```
> yg:=add(c[k]*Y[k], k=1..2);
```

$$yg := c_1 \cos(5\,t) + c_2 \sin(5\,t)$$

```
>
```

Implementation of the initial conditions $y(0) = .3$, $y'(0) = -.1$

```
> eq1:=eval(subs(t=0, yg))=0.3;
  eq2:=eval(subs(t=0, diff(yg, t)))=-0.1;
  val_c:=solve({eq1, eq2}, {c[1], c[2]});
```

$$eq1 := c_1 = 0.3$$

$$eq2 := 5\,c_2 = \text{-}0.1$$

$$val\_c := \{c_1 = 0.3000000000,\ c_2 = \text{-}0.02000000000\}$$

The equation of motion for this undamped vibrating spring is given by

```
> ans[`32a`]:=y=subs(val_c, yg);
```

$$ans_{32a} := y = 0.3000000000\cos(5\,t) - 0.02000000000\sin(5\,t)$$

```
>
```

32b

The frequency of the oscillation equals

$$\frac{\beta}{2\pi} = \frac{5}{2\pi}$$

>

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33a

We solve the initial value problem $10\,y'' + 60\,y' + 250\,y = 0$, $y(0) = .3$, $y'(0) = -.1$

The general solution of the differential equation

Code the differential equation and compute the auxiliary equation.

```
> deq:=10*diff(y(t),t$2)+60*diff(y(t), t)+250*y(t)=0;
```

$$deq := 10\left(\frac{d^2}{dt^2}\,y(t)\right) + 60\left(\frac{d}{dt}\,y(t)\right) + 250\,y(t) = 0$$

```
> aux_eq:=simplify(eval(subs(y(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := 10\,r^2 + 60\,r + 250 = 0$$

Solve the auxiliary equation and find the eigenvalues.

```
> evals:=solve(aux_eq, r);
```

$$evals := -3 + 4\,I,\ -3 - 4\,I$$

Observe that the complex eigenvalues now have a nonzero real part. Hence, the solutions are exponentially damped oscillations.

```
> Y[1]:=exp(Re(evals[1])*t)*cos(Im(evals[1])*t);
  Y[2]:=exp(Re(evals[1])*t)*sin(Im(evals[1])*t);
```

$$Y_1 := \mathbf{e}^{(-3\,t)}\cos(4\,t)$$

$$Y_2 := \mathbf{e}^{(-3\,t)}\sin(4\,t)$$

The general solution is given by an arbitrary linear combination of Y_1 and Y_2.

```
> yg:=add(c[k]*Y[k], k=1..2);
```

$$yg := c_1\,\mathbf{e}^{(-3\,t)}\cos(4\,t) + c_2\,\mathbf{e}^{(-3\,t)}\sin(4\,t)$$

>

Implementation of the initial conditions $y(0) = .3$, $y'(0) = -.1$

```
> eq1:=eval(subs(t=0, yg))=0.3;
  eq2:=eval(subs(t=0, diff(yg, t)))=-0.1;
  val_c:=solve({eq1, eq2}, {c[1], c[2]});
```

$$eq1 := c_1 = 0.3$$

$$eq2 := -3\,c_1 + 4\,c_2 = -0.1$$

$$val\_c := \{c_1 = 0.3000000000,\ c_2 = 0.2000000000\}$$

The equation of motion for this undamped vibrating spring is given by

```
> ans[`32a`]:=y=subs(val_c, yg);
```

$$ans_{32a} := y = 0.3000000000\,\mathbf{e}^{(-3\,t)}\cos(4\,t) + 0.2000000000\,\mathbf{e}^{(-3\,t)}\sin(4\,t)$$

>

33b

The frequency of the oscillation equals

$$\frac{\beta}{2\pi} = \frac{4}{2\pi} = \frac{2}{\pi}$$

33c

Clearly, the damping has caused the frequency of the oscillation to diminish. In addition the oscillation is now exponentially damped, so

$$\lim_{t\to\infty} y(t) = 0$$

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Observe that, since $\frac{dq}{dt} = I$, differentiation of both sides of the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E$$

will lead to

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

and if the elektromotive force E is a constant, then $\frac{dE}{dt} = 0$, so the last equation becomes homogeneous.

\>

The solution of the homogeneous equation.

Because Maple uses the letter I for the imaginary unit, we will denote the current at time t by i(t).

```
> pars:={L=10, R=20, C=1/6260};
  deq_general:=L*diff(i(t), t$2)+R*diff(i(t), t)+i(t)/C=0;
  deq:=subs(pars, deq_general);
```

$$pars := \{L = 10, R = 20, C = \frac{1}{6260}\}$$

$$deq\_general := L\left(\frac{d^2}{dt^2}\mathrm{i}(t)\right) + R\left(\frac{d}{dt}\mathrm{i}(t)\right) + \frac{\mathrm{i}(t)}{C} = 0$$

$$deq := 10\left(\frac{d^2}{dt^2}\mathrm{i}(t)\right) + 20\left(\frac{d}{dt}\mathrm{i}(t)\right) + 6260\,\mathrm{i}(t) = 0$$

Find the auxiliary equation and compute the eigenvalues.

```
> aux_eq:=simplify(eval(subs(i(t)=exp(r*t), deq))/exp(r*t));
```

$$aux\_eq := 10\,r^2 + 20\,r + 6260 = 0$$

```
> evals:=solve(aux_eq, r);
```

$$evals := -1 + 25\,I, -1 - 25\,I$$

Code the corresponding solutions and generate the general solution of the homogeneous equation.

```
> Y[1]:=exp(Re(evals[1])*t)*cos(Im(evals[1])*t);
  Y[2]:=exp(Re(evals[1])*t)*sin(Im(evals[1])*t);
```

$$Y_1 := \mathbf{e}^{(-t)}\cos(25\,t)$$

$$Y_2 := \mathbf{e}^{(-t)}\sin(25\,t)$$

```
> ig:=add(c[k]*Y[k], k=1..2);
```

$$ig := c_1\,\mathbf{e}^{(-t)}\cos(25\,t) + c_2\,\mathbf{e}^{(-t)}\sin(25\,t)$$

\>

Implementation of the initial conditions

Since it is given that the current at time zero is equal to zero, the first initial condition is

$$\mathrm{i}(0) = 0$$

To find a value for $i\,'(0)$ we have to do a little more work. Note it is given that $\mathrm{I}(0) = 0$, $\mathrm{q}(0) = 0$ and $\mathrm{E}(t) = 100$. We can substitute these values in equation (20)

$$(20) \qquad L\frac{dI}{dt} + RI + \frac{1}{C}q = E$$

and find

$$10 \left. \frac{dI}{dt} \right|_{t=0} = 100$$

so

$$i'(0) = 10$$

```
> eq1:=eval(subs(t=0, ig))=0;
  eq2:=eval(subs(t=0, diff(ig, t)))=10;
  val_c:=solve({eq1, eq2}, {c[1], c[2]});
```

$$eq1 := c_1 = 0$$

$$eq2 := -c_1 + 25\, c_2 = 10$$

$$val\_c := \{ c_1 = 0, c_2 = \frac{2}{5} \}$$

```
> ans[34]:=i(t)=subs(val_c, ig);
```

$$ans_{34} := \mathrm{i}(t) = \frac{2}{5} \mathbf{e}^{(-t)} \sin(25\, t)$$

```
>
```