

Lesson 4

Area Approximations

▼ Initializations

```
> restart;  
with(student):  
>
```

▼ 4.1 Area approximations using Left, Right, and Middle Sums.

▼ Examples

▼ Example 4.1.1

Let A denote the area of the region enclosed by the curves

$$y = x^2, x = 1, x = 3, \text{ and } y = 0$$

Do the following:

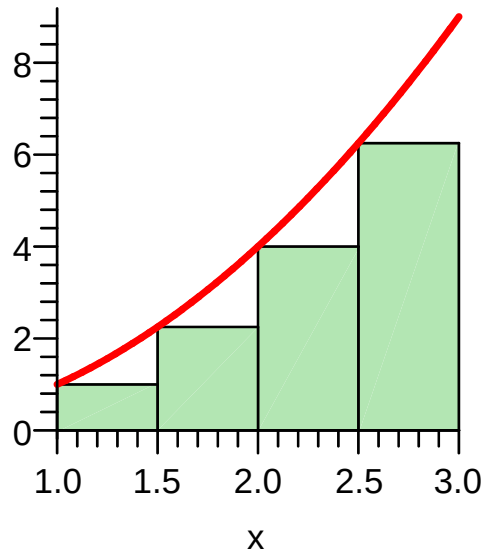
- Approximate the value of A by a left sum of four elements.
- Approximate the value of A by a left sum of twenty elements.
- Approximate the value of A by a left sum of n elements.
- Find the exact value of A by taking the limit as $n \rightarrow \infty$ of the result of Part iii.

Solution

- Approximate the value of A by a left sum of four elements.

Create a picture using Maple's `leftbox` command.

```
> f:=x->x^2;  
leftbox(f(x), x=1..3, 4);  
f:=x→x2
```



Generate the left sum using Maple's **leftsum** command.

```
> L4:=leftsum(f(x), x=1..3, 4);
vL4:=evalf(L4);
```

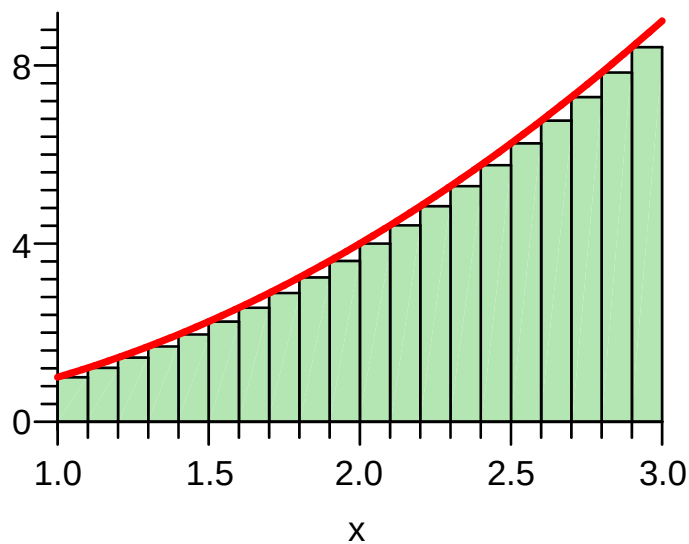
$$L4 := \frac{1}{2} \left(\sum_{i=0}^3 \left(1 + \frac{1}{2} i \right)^2 \right)$$

$vL4 := 6.750000000$

(2.1.1.1)

ii) Approximate the value of A by a left sum of twenty elements.
Repeat the procedure with twenty sub-intervals.

```
> leftbox(f(x), x=1..3, 20);
```



```
> L20:=leftsum(f(x), x=1..3, 20);
vL20:=evalf(L20);
```

$$L20 := \frac{1}{10} \left(\sum_{i=0}^{19} \left(1 + \frac{1}{10} i \right)^2 \right)$$

$$vL20 := 8.270000000 \quad (2.1.1.2)$$

iii) Approximate the value of A by a left sum of n elements.

Once more repeat the same procedure, this time with n sub-intervals.

```
> Ln:=leftsum(f(x), x=1..3, n);
vLn:=simplify(value(Ln));
```

$$Ln := \frac{2 \left(\sum_{i=0}^{n-1} \left(1 + \frac{2i}{n} \right)^2 \right)}{n}$$

$$vLn := \frac{2}{3} \frac{13 n^2 - 12 n + 2}{n^2} \quad (2.1.1.3)$$

iv) Find the exact value of A by taking the limit as $n \rightarrow \infty$ of the result of Part iii.

To find the exact value of the area, we evaluate

$$\lim_{n \rightarrow \infty} \frac{2}{3} \frac{13 n^2 - 12 n + 2}{n^2}$$

```
> L:=Limit(vLn, n=infinity);
```

$$L := \lim_{n \rightarrow \infty} \left(\frac{2}{3} \frac{13 n^2 - 12 n + 2}{n^2} \right) \quad (2.1.1.4)$$

```
> area:=value(L);
evalf(area);
```

$$area := \frac{26}{3}$$

$$8.666666667 \quad (2.1.1.5)$$

```
>
```

Example 4.1.2

Let A denote the area of the region enclosed by the curves

$$y = x^3 - 4x^2 + 2x + 7, x = 0, x = 4, \text{ and } y = 0$$

Do the following:

- Approximate the value of A by a middle sum of four elements.
- Approximate the value of A by a middle sum of twenty elements.
- Approximate the value of A by a middle sum of n elements.
- Find the exact value of A by taking the limit as $n \rightarrow \infty$ of the result of Part iii.

Solution

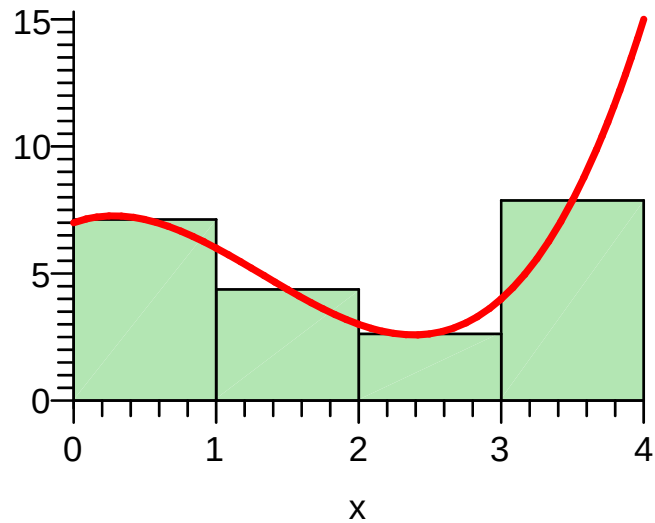
i) Approximate the value of A by a middle sum of four elements.

Create a picture using Maple's **leftbox** command.

```
> f:=x->x^3-4*x^2+2*x+7;
```

```
middlebox(f(x), x=0..4, 4);
```

$$f := x \rightarrow x^3 - 4x^2 + 2x + 7$$



Generate the middle sum using Maple's **middlesum** command.

```
> M4:=middlesum(f(x), x=0..4, 4);
vM4:=evalf(M4);
```

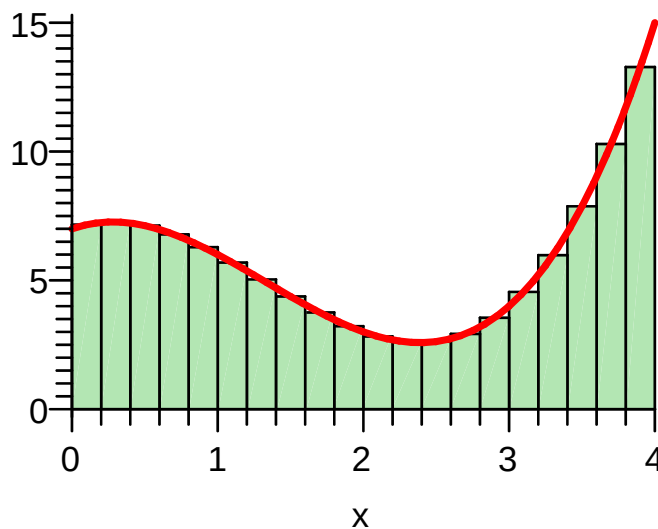
$$M4 := \sum_{i=0}^3 \left(\left(i + \frac{1}{2} \right)^3 - 4 \left(i + \frac{1}{2} \right)^2 + 2i + 8 \right)$$

$$vM4 := 22.00000000$$

(2.1.2.1)

ii) **Approximate the value of A by a middle sum of twenty elements.**
Repeat the procedure with twenty sub-intervals.

```
> middlebox(f(x), x=0..4, 20);
```



```
> M20:=middlesum(f(x), x=0..4, 20);
vM20:=evalf(M20);
```

$$M20 := \frac{1}{5} \left(\sum_{i=0}^{19} \left(\left(\frac{1}{5} i + \frac{1}{10} \right)^3 - 4 \left(\frac{1}{5} i + \frac{1}{10} \right)^2 + \frac{2}{5} i + \frac{36}{5} \right) \right)$$

$$vM20 := 22.64000000 \quad (2.1.2.2)$$

iii) Approximate the value of A by a middle sum of n elements.

Once more repeat the same procedure, this time with n sub-intervals.

> Mn:=middlesum(f(x), x=0..4, n);

vMn:=simplify(value(Mn));

$$Mn := \frac{4 \left(\sum_{i=0}^{n-1} \left(\frac{64 \left(i + \frac{1}{2} \right)^3}{n^3} - \frac{64 \left(i + \frac{1}{2} \right)^2}{n^2} + \frac{8 \left(i + \frac{1}{2} \right)}{n} + 7 \right) \right)}{n}$$

$$vMn := \frac{4}{3} \frac{17 n^2 - 8}{n^2} \quad (2.1.2.3)$$

iv) Find the exact value of A by taking the limit as $n \rightarrow \infty$ of the result of Part iii.

To find the exact value of the area, we evaluate

$$\lim_{n \rightarrow \infty} \frac{4}{3} \frac{17 n^2 - 8}{n^2}$$

> M:=Limit(vMn, n=infinity);

$$M := \lim_{n \rightarrow \infty} \left(\frac{4}{3} \frac{17 n^2 - 8}{n^2} \right) \quad (2.1.2.4)$$

> area:=value(M);

evalf(area);

$$area := \frac{68}{3}$$

$$22.66666667 \quad (2.1.2.5)$$

>