Math 174
Calculus II

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Lesson 10
Integration of Trigonometric Functions

Initializations

```
> restart;
  with(student):
  with(oneonta):
>
```

10.1 Integration of Trigonometric Functions

Examples

Example 10.1.1

Evaluate $\int \sin^4 x \cos^5 x \, dx$.

Solution
Since one of the exponents is odd, so a simple u-substitution will suffice.

```
> e1:=Int(sin(x)^4*cos(x)^5, x);
  rr:=u=sin(x);
```

$$e1 := \int \sin(x)^4 \cos(x)^5 \, dx$$

$$rr := u = \sin(x) \quad \textbf{(2.1.1.1)}$$

```
> e2:=changevar(rr, e1, u);
```

$$e2 := \int u^4 \left(1 - u^2\right)^2 du \quad \textbf{(2.1.1.2)}$$

Let Maple evaluate this elementary integral.

```
> e3:=value(e2)+C;
```

$$e3 := \frac{1}{9} u^9 - \frac{2}{7} u^7 + \frac{1}{5} u^5 + C \quad \textbf{(2.1.1.3)}$$

Substitute $u = \sin x$ into the expression above to obtain the answer in terms of the variable x.

```
> e4:=subs(rr, e3);
```

$$e4 := \frac{1}{9} \sin(x)^9 - \frac{2}{7} \sin(x)^7 + \frac{1}{5} \sin(x)^5 + C \quad \textbf{(2.1.1.4)}$$

```
>
```

Example 10.1.2

Evaluate $\int \tan^6 x \sec^4 x\, dx$.

Solution
Since the power of the secant is even, the substitution $u = \tan x$ will solve the problem.

```
> e1:=Int(tan(x)^6*sec(x)^4, x);
  rr:=u=tan(x);
```

$$e1 := \int \tan(x)^6 \sec(x)^4 \, dx$$

$$rr := u = \tan(x) \quad \textbf{(2.1.2.1)}$$

```
> e2:=changevar(rr, e1, u);
```

$$e2 := \int u^6 \left(1 + u^2\right) du \quad \textbf{(2.1.2.2)}$$

Let Maple evaluate this elementary integral.

```
> e3:=value(e2)+C;
```

$$e3 := \frac{1}{9} u^9 + \frac{1}{7} u^7 + C \quad \textbf{(2.1.2.3)}$$

Substitute $u = \tan x$ into the expression above to obtain the answer in terms of the variable x.

```
> e4:=subs(rr, e3);
```

$$e4 := \frac{1}{9} \tan(x)^9 + \frac{1}{7} \tan(x)^7 + C \quad \textbf{(2.1.2.4)}$$

```
>
```

Example 10.1.3

Evaluate $\int \tan^3 x \sec^5 x\, dx$.

Solution
Since the power of the tangent is odd and the power of the secant is odd, we have to rely on the substitution $u = \sec x$ to solve the problem.

```
> e1:=Int(tan(x)^3*sec(x)^5, x);
  rr:=u=sec(x);
```

$$e1 := \int \tan(x)^3 \sec(x)^5 \, dx$$

$$rr := u = \sec(x) \quad \textbf{(2.1.3.1)}$$

```
> e2:=changevar(rr, e1, u);
```

$$e2 := \int u^6 \left(1 - \frac{1}{u^2}\right) du \quad \textbf{(2.1.3.2)}$$

Let Maple evaluate this elementary integral.

```
> e3:=value(e2)+C;
```

$$e3 := \frac{1}{7} u^7 - \frac{1}{5} u^5 + C \quad \textbf{(2.1.3.3)}$$

Substitute $u = \sec x$ into the expression above to obtain the answer in terms of the variable x.

```
> e4:=subs(rr, e3);
```

$$e4 := \frac{1}{7}\sec(x)^7 - \frac{1}{5}\sec(x)^5 + C \qquad (2.1.3.4)$$

>

Example 10.1.4

Evaluate $\int \sin^6 4x\, dx$.

Solution

The power of the sine is even, so we have to use a double angle formula. The necessary routine can be found in the **oneonta** package.

```
> e1:=Int(sin(4*x)^6, x);
```

$$e1 := \int \sin(4x)^6\, dx \qquad (2.1.4.1)$$

```
> e2:=subs(doubleangle(4, 6, x), e1);
```

$$e2 := \int \left(\frac{1}{2} - \frac{1}{2}\cos(8x)\right)^3 dx \qquad (2.1.4.2)$$

```
> e3:=expand(e2);
```

$$e3 := \frac{1}{8}\int 1\, dx - \frac{3}{8}\int \cos(8x)\, dx + \frac{3}{8}\int \cos(8x)^2\, dx - \frac{1}{8}\int \cos(8x)^3\, dx \qquad (2.1.4.3)$$

Since there still is one integral with an even power of the cosine function, we apply the **doubleangle** routine once more.

```
> e4:=subs(doubleangle(8, 2, x), e3);
```

$$e4 := \frac{1}{8}\int 1\, dx - \frac{3}{8}\int \cos(8x)\, dx + \frac{3}{8}\int \left(\frac{1}{2} + \frac{1}{2}\cos(16x)\right) dx - \frac{1}{8}\int \cos(8x)^3\, dx \qquad (2.1.4.4)$$

Now use Maple's integration routine to evaluate this elementary integral.

```
> e5:=value(e4)+C;
```

$$e5 := \frac{5}{16}x - \frac{11}{192}\sin(8x) + \frac{3}{256}\sin(16x) - \frac{1}{192}\cos(8x)^2\sin(8x) + C \qquad (2.1.4.5)$$

Note that equivalent trigonometric expressions can appear quite different, for example application of Maple's integration routine to the original integral, yields

```
> e6:=value(e1)+C;
```

$$e6 := -\frac{1}{24}\sin(4x)^5\cos(4x) - \frac{5}{96}\sin(4x)^3\cos(4x) - \frac{5}{64}\cos(4x)\sin(4x) + \frac{5}{16}x + C \qquad (2.1.4.6)$$

To show that the expressions e5 and e6 are equivalent, use the combine command.

```
> check:=combine(e5-e6, trig);
```

$$check := 0 \qquad (2.1.4.7)$$

>