Math 174
Calculus II

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Lesson 16
Arc Length

Initializations

```
> restart;
  with(student):
```

16.1 Arc Length

Examples

The arc length L of the curve $y = f(x)$, $a \le x \le b$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Mathematical details will be provided in class.

Example 16.1.1

Compute the arc length of the curve $f(x) = \frac{x^3}{2} + \frac{1}{2x}$, $1 \le x \le 2$.

Solution
Use the formula for arc length given at the top of this worksheet.

```
> f:=x->x^3/6+1/(2*x);
```

$$f := x \to \frac{1}{6}x^3 + \frac{1}{2x} \quad (2.1.1.1)$$

```
> e1:=Int(sqrt(1+D(f)(x)^2), x=1..2);
```

$$e1 := \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2x^2}\right)^2}\, dx \quad (2.1.1.2)$$

```
> e2:=simplify(e1);
```

$$e2 := \frac{1}{2}\int_1^2 \frac{x^4+1}{x^2}\, dx \quad (2.1.1.3)$$

```
> arclength:=value(e2);
  evalf(arclength);
```

$$arclength := \frac{17}{12}$$

$$1.416666667 \quad (2.1.1.4)$$

```
>
```

Example 16.1.2

Find the arc length of the curve $f(x) = \frac{x^2}{2} + \frac{\ln x}{4}$, $2 \le x \le 4$.

Solution

Use the formula for arc length given at the top of this worksheet.

```
> f:=x->x^2/2+ln(x)/4;
```

$$f := x \rightarrow \frac{1}{2}x^2 + \frac{1}{4}\ln(x) \quad (2.1.2.1)$$

```
> e1:=Int(sqrt(1+D(f)(x)^2), x=2..4);
```

$$e1 := \int_2^4 \sqrt{1 + \left(x + \frac{1}{4x}\right)^2}\, dx \quad (2.1.2.2)$$

```
> e2:=simplify(e1);
```

$$e2 := \frac{1}{4}\int_2^4 \frac{\sqrt{24x^2 + 16x^4 + 1}}{x}\, dx \quad (2.1.2.3)$$

It should be clear that this integral can be evaluated by applying the substitution $u = x^2$ followed by a trigonometric substitution. We simply use Maple's power to evaluate this integral.

```
> e3:=value(e2);
```

$$e3 := \frac{1}{8}\sqrt{4481} + \frac{1}{16}\ln\left(\sqrt{4481} - 1\right) - \frac{1}{16}\ln\left(1 + \sqrt{4481}\right) - \frac{1}{16}\ln\left(67\sqrt{4481} + 13443\right) - \frac{3}{16}\ln\left(67\sqrt{4481} - 4481\right) + \frac{1}{16}\ln\left(-67\sqrt{4481} + 13443\right) + \frac{3}{16}\ln\left(67\sqrt{4481} + 4481\right) - \frac{1}{8}\sqrt{353} - \frac{1}{16}\ln\left(\sqrt{353} - 1\right) + \frac{1}{16}\ln\left(1 + \sqrt{353}\right) + \frac{1}{16}\ln\left(19\sqrt{353} + 1059\right) + \frac{3}{16}\ln\left(19\sqrt{353} - 353\right) - \frac{1}{16}\ln\left(-19\sqrt{353} + 1059\right) - \frac{3}{16}\ln\left(19\sqrt{353} + 353\right) \quad (2.1.2.4)$$

Looking at this result, one starts to appreciate the value of a decimal approximation.

```
> arclength_f:=evalf(e3);
```

$$arclength\_f := 6.498799194 \quad (2.1.2.5)$$

Alternatively this result can be obtained by numeric integration of

$$\int_2^4 \sqrt{1+\left(x+\frac{1}{4x}\right)^2}\,dx.$$

```
> e1;
check:=evalf(e1);
```

$$\int_2^4 \sqrt{1+\left(x+\frac{1}{4\,x}\right)^2}\,\mathrm{d}x$$

$$check := 6.498799179 \qquad (2.1.2.6)$$

```
>
```

Example 16.1.3

Compute the arc length of the curve $f(x) = x \sin x,\ 0 \le x \le 1$.

Solution

Integrals representing arc length, frequently cannot be evaluated exactly. This holds true even for some of the most elementary functions. This example represents one of those cases.

```
> f:=x->x*sin(x);
```

$$f := x \to x \sin(x) \qquad (2.1.3.1)$$

```
> e1:=Int(sqrt(1+D(f)(x)^2), x=0..1);
```

$$e1 := \int_0^1 \sqrt{1+(\sin(x)+x\cos(x))^2}\,\mathrm{d}x \qquad (2.1.3.2)$$

```
> value(e1);
```

$$\int_0^1 \sqrt{1+(\sin(x)+x\cos(x))^2}\,\mathrm{d}x \qquad (2.1.3.3)$$

Maple leaves the integral unevaluated. Numeric integration offers the only solution. First, we use Maple's default integration routine.

```
> L:=evalf(e1);
```

$$L := 1.351336280 \qquad (2.1.3.4)$$

Then we apply something as simple as the Simpson's rule with 20 sub-intervals.

```
> e2:=simpson(sqrt(1+D(f)(x)^2), x=0..1, 20);
```

$$e2 := \frac{1}{60} + \frac{1}{60}\sqrt{1+(\sin(1)+\cos(1))^2} + \frac{1}{15}\left(\sum_{i=1}^{10} \sqrt{1+\left(\sin\left(\frac{1}{10}\,i-\frac{1}{20}\right)+\left(\frac{1}{10}\,i-\frac{1}{20}\right)\cos\left(\frac{1}{10}\,i-\frac{1}{20}\right)\right)^2}\right) + \frac{1}{30}\left(\sum_{i=1}^{9}\sqrt{1+\left(\sin\left(\frac{1}{10}\,i\right)+\frac{1}{10}\,i\cos\left(\frac{1}{10}\,i\right)\right)^2}\right) \qquad (2.1.3.5)$$

```
> S:=evalf(e2);
```

(2.1.3.6)

$$S := 1.351336227 \tag{2.1.3.6}$$

The results L and S are almost indistinguishable.

>