Math 276
Calculus III

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Lesson 3

Vector Arithmetic, Dot Products and Angles between Vectors

Initializations

```
> restart;
  with(VectorCalculus):
  BasisFormat(false):
>
```

3.1 Vector Arithmetic

Examples

Example 3.1.1

Let **a** = ⟨1, 2,–3⟩ and **b** = ⟨–1, 2, 5⟩. Compute **a** + **b**, **a** − **b**, 2**a** − 3**b**, |**a**|, and a unit vector in the direction of **b**.

Solution
The syntax will speak for itself.

```
> a:=<1, 2, -3>;
  b:=<-1, 2, 5>;
```

$$a := \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$b := \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \quad (2.1.1.1)$$

```
> ans1:=a+b;
```

$$ans1 := \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \quad (2.1.1.2)$$

```
> ans2:=a-b;
```

(2.1.1.3)

$$ans2 := \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix} \tag{2.1.1.3}$$

```
> ans3:=2*a-3*b;
```

$$ans3 := \begin{bmatrix} 5 \\ -2 \\ -21 \end{bmatrix} \tag{2.1.1.4}$$

To compute the length of the vector **a**, we use the **Norm** command. The syntax is Norm(**a**);

```
> ans4:=Norm(a);
```

$$ans4 := \sqrt{14} \tag{2.1.1.5}$$

To compute the unit vector in the direction of the vector **b**, we divide **b** by its norm.

```
> ans5:=b/Norm(b);
```

$$ans5 := \begin{bmatrix} -\frac{1}{30}\sqrt{30} \\ \frac{1}{15}\sqrt{30} \\ \frac{1}{6}\sqrt{30} \end{bmatrix} \tag{2.1.1.6}$$

3.2 Dot products and angles between vectors

Examples

Example 3.2.1

Find the dot product of the vectors **a** = $\langle u, v, w \rangle$, and **b** = $\langle x, y, z \rangle$.

Solution

Code the vectors and use the **DotProduct** command.

```
> a:=<u, v, w>;
  b:=<x, y, z>;
```

$$a := \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$b := \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.1.1.1}$$

```
> dp:=DotProduct(a, b);
```

$$dp := u\,x + v\,y + w\,z \tag{3.1.1.2}$$

Note: For coding enthusiasts is may be nice to know that the code

$$\mathbf{a} \cdot \mathbf{b}$$

also computes the dot product of the vectors **a** and **b**.

```
> a.b;
```

$$u\,x + v\,y + w\,z \tag{3.1.1.3}$$

```
>
```

Example 3.2.2

Find the angle between the vectors $\mathbf{a} = \langle 1, 2, -7 \rangle$ and $\mathbf{b} = \langle 11, -2, -8 \rangle$.

Solution

Code the vectors

```
> a:=<1, 2, -7>;
  b:=<11, -2, -8>;
```

$$a := \begin{bmatrix} 1 \\ 2 \\ -7 \end{bmatrix}$$

$$b := \begin{bmatrix} 11 \\ -2 \\ -8 \end{bmatrix} \tag{3.1.2.1}$$

Compute the cosine of the angle between, **a** and **b**, using the formula derived in-class.

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

It pays to simplify your result.

```
> cosine_theta:=simplify(DotProduct(a, b)/(Norm(a)*Norm(b)));
```

$$cosine\_theta := \frac{1}{6}\sqrt{2}\sqrt{7} \tag{3.1.2.2}$$

Find the angle itself by using the arc-cosine function.

```
> ans:=arccos(cosine_theta);
```

$$ans := \arccos\left(\frac{1}{6}\sqrt{2}\sqrt{7}\right) \tag{3.1.2.3}$$

It pays to use a decimal approximation.

```
> ansf:=evalf(ans);
```

$$ansf := 0.8974447095 \tag{3.1.2.4}$$

Of course this answer is given in radians. We convert the result to degrees.

```
> ansdf:=evalf(ansf*180/Pi);
```

$$ansdf := 51.41979418 \tag{3.1.2.5}$$

```
>
```

Example 3.2.3

Find the directional cosines of the vector $\langle 1, 5, -7\rangle$.

Solution

Code the vector and compute a unit vector in the same direction. The components of that unit vector are the desired directional cosines.

```
> v:=<1, 5, -7>;
```

$$v := \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix} \qquad (3.1.3.1)$$

```
> u_v:=v/Norm(v);
```

$$u\_v := \begin{bmatrix} \frac{1}{15}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ -\frac{7}{15}\sqrt{3} \end{bmatrix} \qquad (3.1.3.2)$$

```
>
```

Example 3.2.4

Find the scalar projection (the component) and the vector projection of $\mathbf{b} = \langle 1, 1, 2\rangle$ onto $\mathbf{a} = \langle -2, 3, 1\rangle$.

Solution

Use the standard formulas for these projections.

$$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} \quad \text{and} \quad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\,\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$$

```
> a:=<-2, 3, 1>;
  b:=<1, 1, 2>;
```

$$a := \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$b := \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad (3.1.4.1)$$

```
> sc_p:=DotProduct(a, b)/Norm(a);
```

(3.1.4.2)

$$sc\_p := \frac{3}{14}\sqrt{14} \quad (3.1.4.2)$$

```
> vec_p:=sc_p*a/Norm(a);
```

$$vec\_p := \begin{bmatrix} -\frac{3}{7} \\ \frac{9}{14} \\ \frac{3}{14} \end{bmatrix} \quad (3.1.4.3)$$

```
>
```

Example 3.2.5

The force **F** = ⟨3, 4, 5⟩ moves a particle from the point $P = (2, 1, 0)$ to the point $Q = (4, 6, 2)$. Find the work done by the force.

Solution

Code the force vector **F** and the position vectors of the points.

```
> F:=<3, 4, 5>;
  P:=<2, 1, 0>;
  Q:=<4, 6, 2>;
```

$$F := \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$P := \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$Q := \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} \quad (3.1.5.1)$$

The work done equals the dot product of the force and the displacement.

```
> Work:=DotProduct(F, Q-P);
```

$$Work := 36 \quad (3.1.5.2)$$

```
>
```