Math 173
Calculus I

Dr. Constant J. Goutziers
Department of Mathematics, Computer Science and Statistics
goutzicj@oneonta.edu

Lesson 4

The Limit of a Function

Initializations

```
> restart;
  with(oneonta):
>
```

4.1 Limits of Functions Defined by One Expression

Examples

Example 4.1.1

Use three different ways to investigate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Solution

First we will use the **values** command in the **oneonta** package.

```
> f:=x->sin(x)/x;
```

$$f := x \rightarrow \frac{\sin(x)}{x} \qquad \textbf{(2.1.1.1)}$$

```
> values(f, x, 0);
```

$$\lim_{x \to 0} f = \lim_{x \to 0} \left(\frac{\sin(x)}{x} \right)$$

Values to the left of the limit point.

x	f
-0.1000000000	0.9983341665
-0.01000000000	0.9999833334
-0.001000000000	0.9999998333
-0.0001000000000	0.9999999983
-0.00001000000000	1.000000000

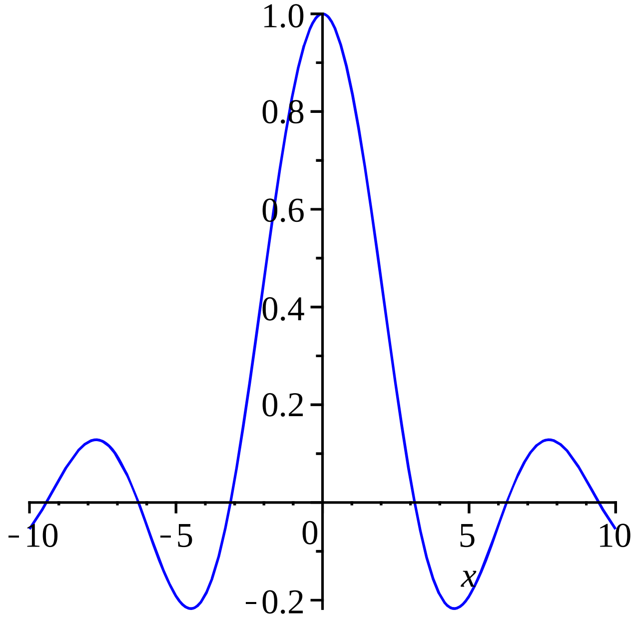
Values to the right of the limit point.

x	f
0.1000000000	0.9983341665
0.01000000000	0.9999833334
0.001000000000	0.9999998333
0.0001000000000	0.9999999983
0.00001000000000	1.000000000

(2.1.1.2)

From the table it appears that this limit equals 1. This result is supported by the graph of $\frac{\sin(x)}{x}$.

```
> plot(f(x), x=-10..10, color=blue);
```



The actual proof of the fact that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ makes use of the squeeze theorem. This will be addressed in class.

Maple does have a routine for evaluating limits.

```
> limit(f(x), x=0);
```

$$1 \qquad \mathbf{(2.1.1.3)}$$

The **Limit** command is the inert version of **limit**, it pretty prints the limit on the screen. The pretty printed version can be evaluated using the value command.

```
> e1:=Limit(f(x), x=0);
```

$$e1 := \lim_{x \to 0} \left(\frac{\sin(x)}{x} \right) \qquad \mathbf{(2.1.1.4)}$$

```
> e2:=value(e1);
```

$$e2 := 1 \qquad (2.1.1.5)$$

4.2 Limits of Functions Defined by Multiple Expressions, One Sided Limits.

Examples

Example 4.2.1

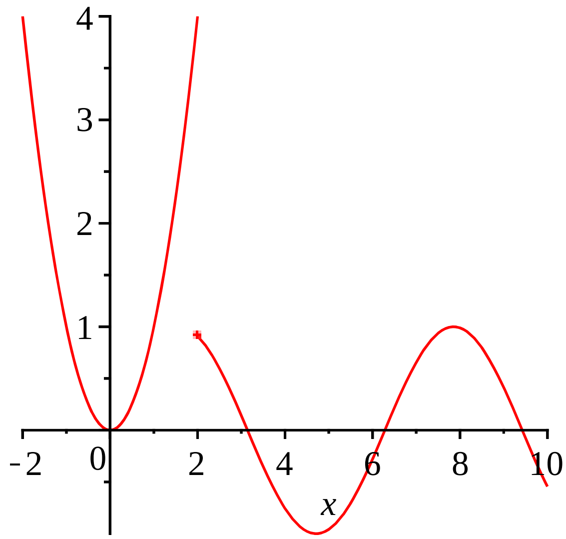
Plot a graph of the function $f(x) = \begin{cases} x^2 & x < 2 \\ \sin(x) & 2 \le x \end{cases}$ **and investigate the one-sided limits of** $f(x)$ **at** $x = 2$**.**

Solution
First we have to define the function *f*. This is done by using Maple's piecewise command. To make sure that the plotting routine will appropriately handle any discontinuities we include the **discont = true** option.

```
> f:=x->piecewise(x<2, x^2, x>=2, sin(x));
```

$$f := x \rightarrow piecewise(x < 2, x^2, 2 \le x, \sin(x)) \qquad (3.1.1.1)$$

```
> plot(f(x), x=-2..10, discont=true);
```



The picture suggests that $\lim_{x \to 2-} f(x) = 4$ **and** $\lim_{x \to 2+} f(x) = \sin(2)$. We can verify these results by using Maple's **Limit** command.

```
> e1:=Limit('f'(x), x=2, left);
```

$$e1 := \lim_{x \to 2^-} f(x) \qquad (3.1.1.2)$$

```
> e2:=value(e1);
```

$$e2 := 4 \qquad (3.1.1.3)$$

```
> e3:=Limit('f'(x), x=2, right);
```

(3.1.1.4)

$$e3 := \lim_{x \to 2+} f(x) \quad (3.1.1.4)$$

```
> e4:=value(e3);
```

$$e4 := \sin(2) \quad (3.1.1.5)$$

Alternatively, these limits could have been explored using the **values** command in the **oneonta** package.

```
> values(f, x, 2);
```

$$\lim_{x \to 2} f = \lim_{x \to 2} \left(\begin{cases} x^2 & x < 2 \\ \sin(x) & 2 \le x \end{cases} \right)$$

Values to the left of the limit point.

x	f
1.900000000	3.610000000
1.990000000	3.960100000
1.999000000	3.996001000
1.999900000	3.999600010
1.999990000	3.999960000

Values to the right of the limit point.

x	f
2.100000000	0.8632093666
2.010000000	0.9050905633
2.001000000	0.9088808254
2.000100000	0.9092558076
2.000010000	0.9092932653

(3.1.1.6)

Observe that sin(2) = 0.90929...

```
> evalf(sin(2));
```

$$0.9092974268 \quad (3.1.1.7)$$

```
>
```