Math 173
Calculus I

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Lesson 9

Hyperbolic Trigonometric Functions

Initializations

```
> restart;
```

9.1 Definitions and Graphs

The hyperbolic sine and hyperbolic cosine functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Note: tanh x, coth x, sech x, and csch x are defined similarly to their trigonometric counterparts. For instance

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Because the hyperbolic functions are structurally similar to the ordinary trigonometric functions, the relationships between them are similar too. For instance

$$\cosh^2 x - \sinh^2 x = 1$$

Such relations can be proved simply by using the definitions of sinh x and cosh x. Just watch

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{4 e^x e^{-x}}{4} = 1$$

In spite of all the similarities, these functions are vastly different from ordinary trigonometric functions. In particular, they are **not** periodic. In the examples we explore some of the properties of the hyperbolic functions.

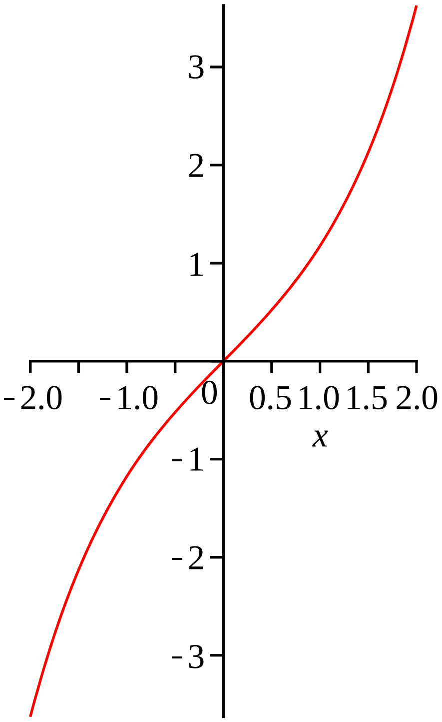
Examples

Example 9.1.1

Plot sinh x and state its domain and range.

Solution

```
> plot(sinh(x), x=-2..2, tickmarks=[10,10], scaling=
  constrained);
```



```
>
```

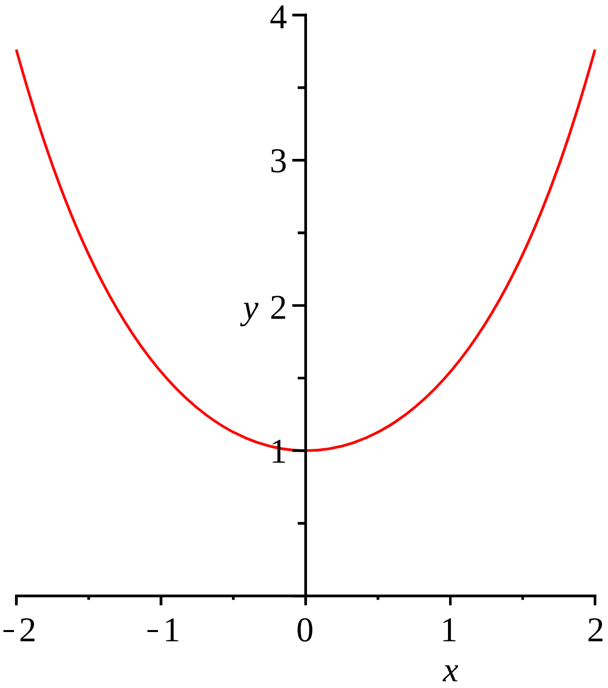
The domain of sinh x is $(-\infty, \infty)$ and the range of sinh x is $(-\infty, \infty)$. The graph of this function has no asymptotes.

Example 9.1.2

Plot cosh x and state its domain and range.

Solution

```
> plot(cosh(x), x=-2..2, y=0..4, scaling=constrained);
```



```
>
```

The domain of $\cosh x$ is $(-\infty, \infty)$ and the range of $\cosh x$ is $[1, \infty)$. Again, the graph of this function has no asymptotes.

Note 1: $\cosh x$ looks like a parabola, it feels like a parabola, **but it is not a parabola!**

Note 2: $\cosh x$ represents a catenary or chain line. Telephone wires and electricity cables hang in the shape of a hyperbolic cosine. The Gateway Arch in St. Louis has the shape of an inverted catenary.

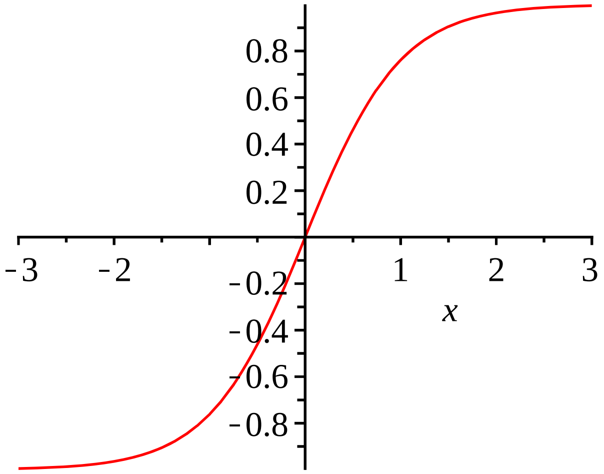
Note 3: The cables of a suspension bridge do **not** hang in the shape of a catenary. Because the weight of the cables dwarfs in comparison with the enormous weight of the road deck, which is uniformly distributed over the length of the bridge, the cables of a suspension bridge actually take the shape of a parabola.

Example 9.1.3

Plot tanh *x* and state its domain and range.

Solution

```
> plot(tanh(x), x=-3..3);
```



```
>
```

The domain of $\tanh x$ is $(-\infty, \infty)$ and the range of $\tanh x$ is $(-1, 1)$. The graph of this function has two horizontal asymptotes, $y = -1$ **and** $y = 1$.

Example 9.1.4

Compute the derivatives of sinh *x*, cosh *x*, and tanh *x*.

Solution

This can be done using the definition of these functions. For instance

$$\frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Similarly

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{d\,x}\tanh x = \operatorname{sech}^2 x = 1 - \tanh^2 x$$

Of course Maple has these derivatives built in.

```
> Diff(sinh(x), x)=diff(sinh(x), x);
```

$$\frac{d}{dx}\sinh(x) = \cosh(x) \tag{2.1.4.1}$$

```
> Diff(cosh(x), x)=diff(cosh(x), x);
```

$$\frac{d}{dx}\cosh(x) = \sinh(x) \tag{2.1.4.2}$$

```
> Diff(tanh(x), x)=diff(tanh(x), x);
```

$$\frac{d}{dx}\tanh(x) = 1 - \tanh(x)^2 \tag{2.1.4.3}$$

```
>
```

9.2 Inverse Hyperbolic Functions

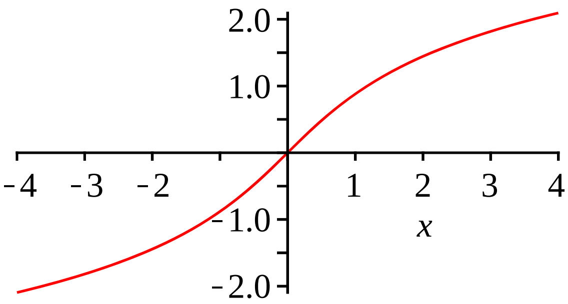
Examples

Example 9.2.1

Plot arcsinh *x* and state its domain and range.

Solution

```
> plot(arcsinh(x), x=-4..4, scaling=constrained, tickmarks=
  [10,10]);
```



```
>
```

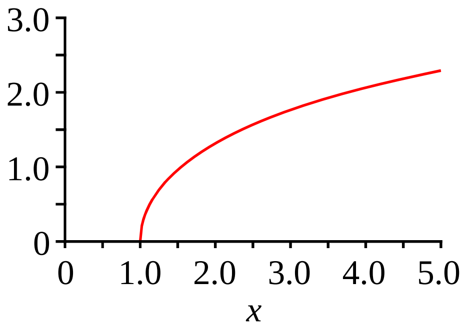
The domain of arcsinh x is $(-\infty, \infty)$ and the range of arcsinh x is $(-\infty, \infty)$. The graph of this function has no asymptotes.

Example 9.2.2

Plot arccosh *x* and state its domain and range.

Solution

```
> plot(arccosh(x), x=1..5, scaling=constrained, tickmarks=
  [10,10], view=[0..5,0..3]);
```



```
>
```

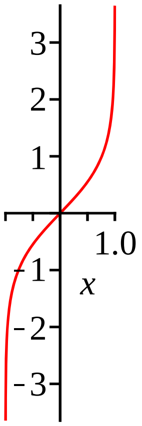
The domain of arccosh x is $[1,\infty)$ and the range of arccosh x is $[0,\ \infty)$. The graph of this function has no asymptotes.

Example 9.2.3

Plot arctanh x and state its domain and range.

Solution

```
> plot(arctanh(x), x=-1..1, scaling=constrained, tickmarks=
  [5,10]);
```



```
>
```

The domain of arctanh x is $(-1,\ 1)$ and the range of arctanh x is $(-\infty,\ \infty)$. The graph of this function has two vertical asymptotes, $x=-1$ **and** $x=1$.

Example 9.2.4

Since hyperbolic functions are essentially exponential functions, it should come as no surprise that the inverse hyperbolic functions are essentially logarithms

$$\operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right),\quad -\infty < x < \infty$$

$$\operatorname{arccosh} x = \ln\left(x+\sqrt{x^2-1}\right),\quad x \geq 1$$

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$$

The mathematical details will be provided in class.

Example 9.2.5

Compute the derivatives of the inverse hyperbolic functions.

Solution

This problem can be solved by differentiating the expressions given in Example 9.2.4.

$$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}, \quad -\infty < x < \infty$$

$$\frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$

$$\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2}, \quad -1 < x < 1$$