Math 174
Calculus II

Dr. Constant J. Goutziers
Department of Mathematical Sciences
goutzicj@oneonta.edu

Lesson 4

Area Approximations

Initializations

```
> restart;
  with(student):
>
```

4.1 Area approximations using Left, Right, and Middle Sums.

Examples

Example 4.1.1

Let A denote the area of the region enclosed by the curves

$$y=x^2,\ x=1,\ x=3,\ \text{and}\ y=0$$

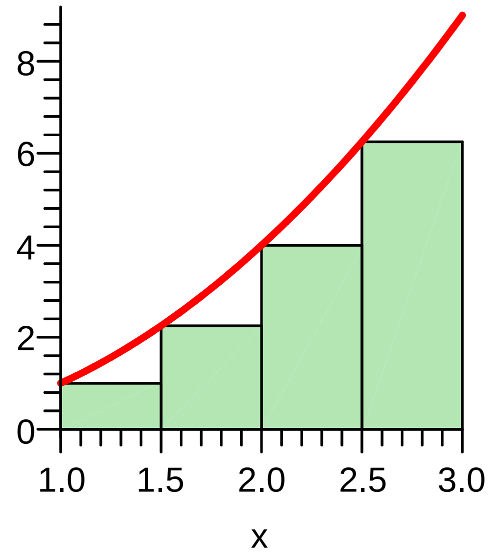
Do the following:

i) Approximate the value of A by a left sum of four elements.
ii) Approximate the value of A by a left sum of twenty elements.
iii) Approximate the value of A by a left sum of n elements.
iv) Find the exact value of A by taking the limit as $n \to \infty$ of the result of Part iii.

Solution
i) Approximate the value of A by a left sum of four elements.
Create a picture using Maple's **leftbox** command.

```
> f:=x->x^2;
  leftbox(f(x), x=1..3, 4);
```

$$f := x \to x^2$$



Generate the left sum using Maple's **leftsum** command.

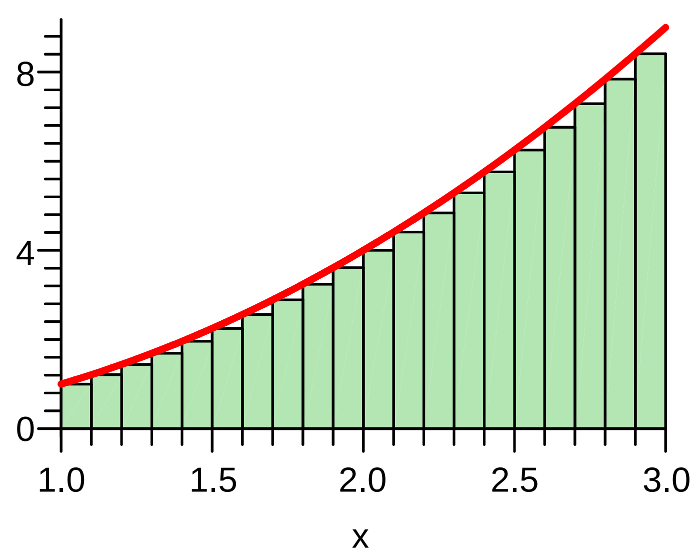
```
> L4:=leftsum(f(x), x=1..3, 4);
  vL4:=evalf(L4);
```

$$L4 := \frac{1}{2}\left(\sum_{i=0}^{3}\left(1+\frac{1}{2}\,i\right)^2\right)$$

$$vL4 := 6.750000000 \qquad (2.1.1.1)$$

ii) Approximate the value of *A* by a left sum of twenty elements.
Repeat the procedure with twenty sub-intervals.

```
> leftbox(f(x), x=1..3, 20);
```




```
> L20:=leftsum(f(x), x=1..3, 20);
  vL20:=evalf(L20);
```

$$L20 := \frac{1}{10}\left(\sum_{i=0}^{19}\left(1+\frac{1}{10}\,i\right)^2\right)$$

$$vL20 := 8.270000000 \qquad (2.1.1.2)$$

iii) Approximate the value of *A* by a left sum of *n* elements.
Once more repeat the same procedure, this time with *n* sub-intervals.

```
> Ln:=leftsum(f(x), x=1..3, n);
  vLn:=simplify(value(Ln));
```

$$Ln := \frac{2\left(\sum_{i=0}^{n-1}\left(1+\frac{2\,i}{n}\right)^2\right)}{n}$$

$$vLn := \frac{2}{3}\,\frac{13\,n^2-12\,n+2}{n^2} \qquad (2.1.1.3)$$

iv) Find the exact value of *A* by taking the limit as $n \to \infty$ of the result of Part iii.
To find the exact value of the area, we evaluate

$$\lim_{n\to\infty}\frac{2}{3}\,\frac{13\,n^2-12\,n+2}{n^2}$$

```
> L:=Limit(vLn, n=infinity);
```

$$L := \lim_{n\to\infty}\left(\frac{2}{3}\,\frac{13\,n^2-12\,n+2}{n^2}\right) \qquad (2.1.1.4)$$

```
> area:=value(L);
  evalf(area);
```

$$area := \frac{26}{3}$$

$$8.666666667 \qquad (2.1.1.5)$$

```
>
```

Example 4.1.2

Let *A* denote the area of the region enclosed by the curves

$$y = x^3 - 4x^2 + 2x + 7,\ x = 0,\ x = 4,\ \text{and}\ y = 0$$

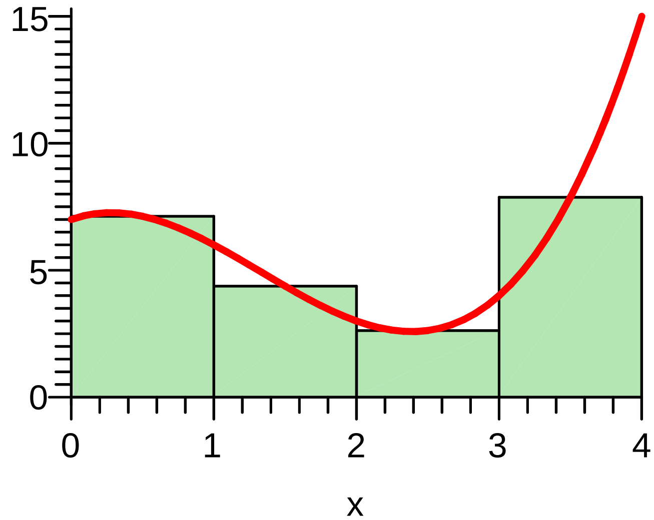
Do the following:

i) Approximate the value of *A* by a middle sum of four elements.
ii) Approximate the value of *A* by a middle sum of twenty elements.
iii) Approximate the value of *A* by a middle sum of *n* elements.
iv) Find the exact value of *A* by taking the limit as $n \to \infty$ of the result of Part iii.

Solution
i) Approximate the value of *A* by a middle sum of four elements.
Create a picture using Maple's **leftbox** command.

```
> f:=x->x^3-4*x^2+2*x+7;
  middlebox(f(x), x=0..4, 4);
```

$$f := x \to x^3 - 4\,x^2 + 2\,x + 7$$



Generate the middle sum using Maple's **middlesum** command.

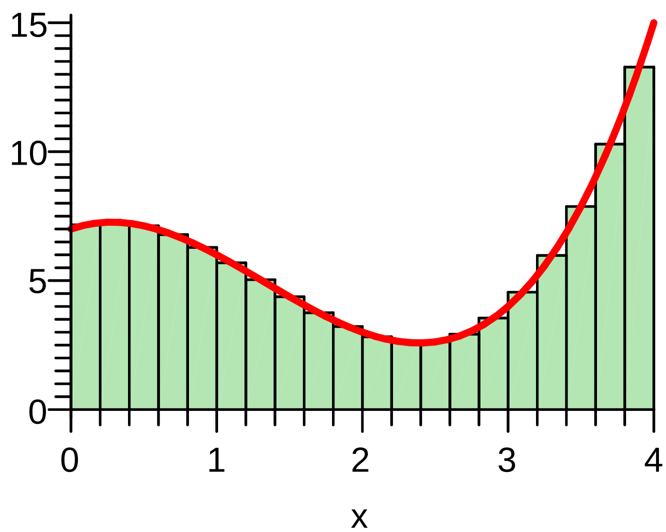
```
> M4:=middlesum(f(x), x=0..4, 4);
  vM4:=evalf(M4);
```

$$M4 := \sum_{i=0}^{3} \left(\left(i+\frac{1}{2}\right)^3 - 4\left(i+\frac{1}{2}\right)^2 + 2\,i + 8 \right)$$

$$vM4 := 22.00000000 \qquad (2.1.2.1)$$

ii) Approximate the value of ***A*** **by a middle sum of twenty elements.**
Repeat the procedure with twenty sub-intervals.

```
> middlebox(f(x), x=0..4, 20);
```




```
> M20:=middlesum(f(x), x=0..4, 20);
  vM20:=evalf(M20);
```

$$M20 := \frac{1}{5}\left(\sum_{i=0}^{19}\left(\left(\frac{1}{5}\,i+\frac{1}{10}\right)^3 - 4\left(\frac{1}{5}\,i+\frac{1}{10}\right)^2 + \frac{2}{5}\,i + \frac{36}{5}\right)\right)$$

$$vM20 := 22.64000000 \tag{2.1.2.2}$$

iii) Approximate the value of *A* by a middle sum of *n* elements.
Once more repeat the same procedure, this time with *n* sub-intervals.

```
> Mn:=middlesum(f(x), x=0..4, n);
  vMn:=simplify(value(Mn));
```

$$Mn := \frac{4\left(\sum_{i=0}^{n-1}\left(\frac{64\left(i+\frac{1}{2}\right)^3}{n^3} - \frac{64\left(i+\frac{1}{2}\right)^2}{n^2} + \frac{8\left(i+\frac{1}{2}\right)}{n} + 7\right)\right)}{n}$$

$$vMn := \frac{4}{3}\,\frac{17\,n^2-8}{n^2} \tag{2.1.2.3}$$

iv) Find the exact value of *A* by taking the limit as *n* → ∞ of the result of Part iii.
To find the exact value of the area, we evaluate

$$\lim_{n\to\infty} \frac{4}{3}\,\frac{17\,n^2-8}{n^2}$$

```
> M:=Limit(vMn, n=infinity);
```

$$M := \lim_{n\to\infty}\left(\frac{4}{3}\,\frac{17\,n^2-8}{n^2}\right) \tag{2.1.2.4}$$

```
> area:=value(M);
  evalf(area);
```

$$area := \frac{68}{3}$$

$$22.66666667 \tag{2.1.2.5}$$

```
>
```