Kater's Pendulum

The Compound Pendulum

A compound pendulum is the term that generally refers to an arbitrary lamina that is allowed to oscillate about a point located some distance from the lamina's center of mass. In such a case if the distance from the point of suspension to the center of mass is known the entire mass of the lamina may be treated as if it were located at the center of mass. If the moment of inertia of the lamina about the point of suspension is known then the equation of motion may be written as:

$$I\ddot{\theta} = -mgl\sin\theta \qquad (1)$$

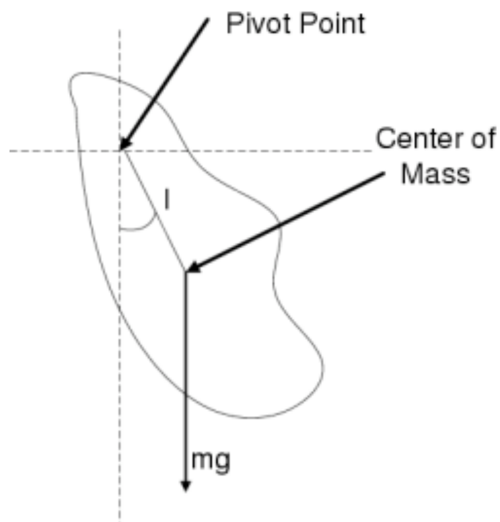


Figure 1

If the angular displacement is considered small then the sine of the angle may be replace by the angle and the moment of inertial about the pivot point may be calculated using the parallel axis theorem so that equation (1) may be written as:

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0 \quad or\ \ddot{\theta} + \omega^2\theta = 0 \qquad (2)$$

$$I = I_{CM} + Ml^2$$

$$\omega^2 = \sqrt{\frac{mgl}{I}}$$

In equation (2) the parallel axis theorem has been used to express the moment of inertial about the pivot point in terms of the moment of inertia about the center of mass and the distance of the point of suspension from the center of mass. Equation (2) is just the equation for a simple harmonic oscillator and has a sinusoidal solution. By analogy the period may be immediately written in term of the angular frequency ω as:

$$\omega = 2\pi f = \frac{2\pi}{T} \quad or \qquad (3)$$

$$T = 2\pi\sqrt{\frac{I}{mgl}}$$

The moment of inertia I may be written in a somewhat more convenient form by defining the quantity k as $I_{cm}= mk^2$ so that: (k is called the radius of gyration and will have a value that depends on the geometry of the lamina.)

$$I = I_{CM} + Ml^2 = m(k^2 + l^2) \qquad (4)$$

When this put into equation (3) the mass cancels and the period may be written as:

$$T = 2\pi\sqrt{\frac{(k^2 + l^2)}{gl}} \qquad (5)$$

If k = 0 the expression for the period then becomes that of a simple pendulum. Can you interpret this limiting case? Hint: What does setting k = 0 imply regarding the moment of inertia about the center of mass?

Kater's Pendulum

The pendulum was a very useful device to determine the acceleration of gravity because many swings of the pendulum could be counted to get an accurate estimate of the period. For precise measurement of g the limiting factor was the determination of the center of mass and the radius of gyration. Kater over came this difficulty by using a pendulum that was allowed to pivot about two points.

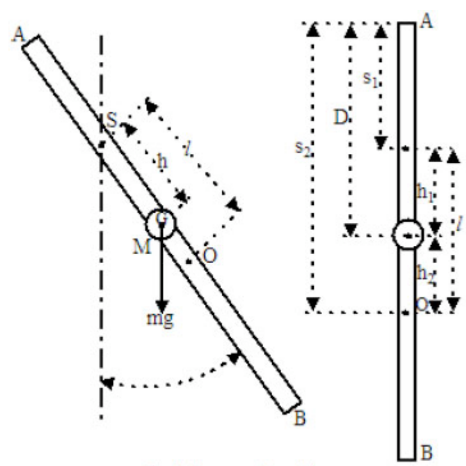


Fig. 1. Compound pendulum

Figure 2

The points $s_1$ and $s_2$ as measured from A are the locations of the two pivot points on the reversible compound pendulum. Letting $s_1 = l_a$ and $s_2 = l_b$ two different periods will be obtained for the oscillation:

$$T_a = 2\pi\sqrt{\frac{(k^2 + l_a^2)}{gl_a}} \qquad (5)$$

$$T_b = 2\pi\sqrt{\frac{(k^2 + l_b^2)}{gl_b}}$$

If the period as a function of $l_a$ or $l_b$ is plotted symmetric curves are obtained. If one is fixed and the other is adjusted it is possible to make the periods identical, so $T_a = T_b$. Then the two expression in (5) can be equated and solved for k, that is:

$$T_a = 2\pi\sqrt{\frac{(k^2 + l_a^2)}{gl_a}} = T_b = 2\pi\sqrt{\frac{(k^2 + l_b^2)}{gl_b}} \qquad (6)$$

$$\frac{(k^2 + l_a^2)}{gl_a} = \frac{(k^2 + l_b^2)}{gl_b} \Rightarrow l_b(k^2 + l_a^2) = l_a(k^2 + l_b^2)$$

$$k^2 = \frac{l_a l_b\left(l_b - l_a\right)}{\left(l_b - l_a\right)} = l_a l_b$$

Putting this back into (5) the period is then given as:

$$T_a = 2\pi\sqrt{\frac{(l_a + l_b)}{g}} \quad or \quad g = \frac{4\pi^2(l_a + l_b)}{T^2} \qquad (7)$$

The beauty of this result is that only the distance between the pivots and the period must be measured thus making it unnecessary to locate the center of mass or determine k. Kater invented the method about 1815.

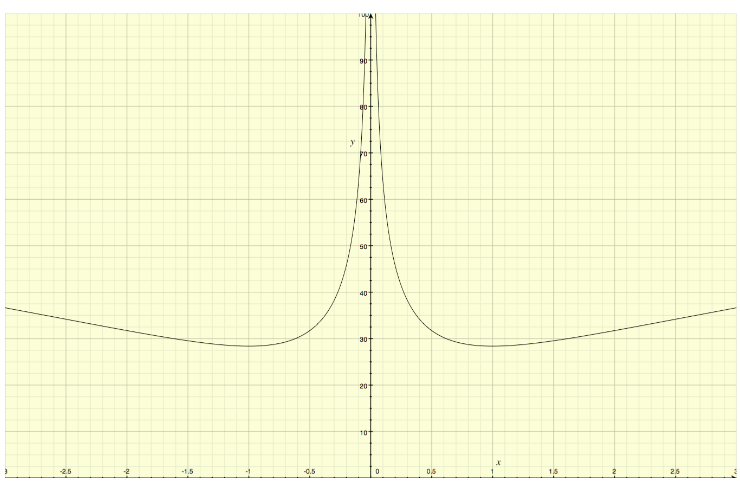

Figure 3

Figure 3 illustrates the period as the pivot point is moved from the center of mass either to the right or left. The vertical axis is time and is a measure of the period and the horizontal axis is the distance from the center of mass.

Application to a Rectangular Lamina

For a rectangle whose width is small compared to its length the moment of inertial about the center of mass is easily calculated. The picture shows a rod

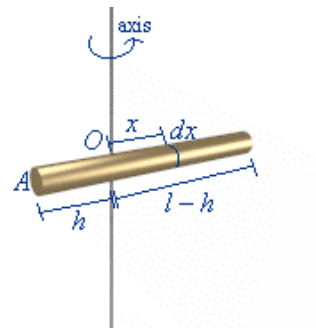


Figure 4

If λ is the mass per unit length then the moment about the center of mass is:

$$I_{CM} = 2\lambda \int_0^{l/2} r^2\, dr = \frac{2\lambda}{3} r^3 \Big|_0^{l/2} = \frac{1}{12} m l^2 \qquad (8)$$

Now using (8) we can calculate the radius of gyration as:

$$mk^2 = \frac{1}{12} m l^2 \Rightarrow k = \frac{l}{\sqrt{12}} \qquad (9)$$

However we also know that for equal periods $l_a \times l_b = k^2$, so that since $l_a = l/2$ it follow that:

$$k^2 = l_a l_b = \frac{l}{2} l_b = \frac{l^2}{12} \;\; or \;\; l_b = \frac{l}{6} \qquad (10)$$

$$l_a + l_b = \frac{2}{3} l$$

So it follows that the period of the pendulum will be the same if it is pivoted from one end and then inverted and pivoted a third of the distance from the other end. Note: I have cheated just a tad by assuming the rod and the rectangular plate have the same moment of inertia about the center of mass, which is a very good approximation provided the rectangular plat has a length >> than its width.

Now what length will give a period of one second? Since we know the accepted value of g we can use either equation for T to find the length. In the calculation g is assumed to be 9.81, so that:

$$T = 2\pi \sqrt{\frac{2l}{3g}} \quad or \quad l = \frac{3gT^2}{8\pi^2} = 37.3cm \qquad (11)$$

Equation 11 implies that by making a Kater's pendulum our of a rod that is 0.373 meters in length it will have a period of 1 second when pivoted about L/2 and L/6. The curve in figure 5 is plotted for a rod that is .3725 meters in length.

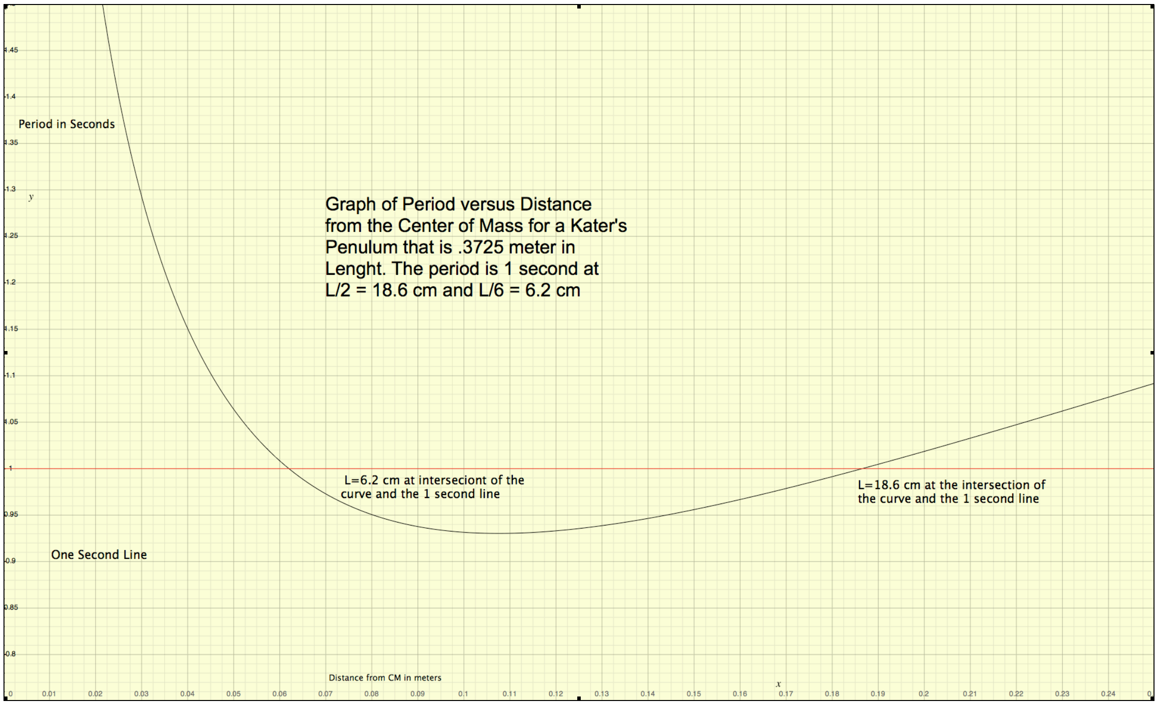


Figure 5

In figure 5 only one half of the graph has been displayed. It should be understood that since the graph in figure 3 is symmetric the same information is displayed. The locations L/2 and L/6 are on opposite sides of the center of mass. As you should understand if the pendulum is pivoted at ±L/2 or ±L/6 it will have a period of one second.