Simple Harmonic Oscillators

Objective

This experiment may be divided into two parts: Determination of the spring constant, and investigation of the period of a simple harmonic oscillator.

Part I

In this section you will determine the spring constant of a spring, which has been wound in such a way that the restoring force is linear. Recall that if the restoring force is linear then the oscillator may be termed simple, which means the period of oscillation is independent of the amplitude of the oscillator. Mounting the spring vertically and recording the elongation of the spring as it is loaded with various masses allows one to collect data for elongation verses applied force and thus the spring constant may be determined.

Procedure

1. Mount the spring vertically with the larger end upward.
2. Using a meter stick and a convenient reference point near the bottom of the spring where the spring is connected to the driving mass
3. Load the spring with 100 grams of mass and record the meter stick reading. Continue this process increasing the load in increments of 50 grams. Obtain a minimum of 5 points. Do not overload the spring (more than 600 grams) as it will cause a deviation from linearity.
4. Plot the value of F = mg versus x the displacement. (F is plotted on the vertical axis and x on the horizontal axis. The slope of the resulting line is k, the spring constant measured in Newtons/meter. (Note: Hooke's law states F=-kx but we plot F=kx only as a matter of convenience so the graph will be in quadrant one.)

Mass / Force	Equilibrium Position	Recorded Position	Displacement
0.2 kg / 1.96 N	.778 meters	0.582 meters	0.196 meters
0.4 kg / 3.93 N	.778 meters	0.381 meters	0.397 meters
0.5 kg / 4.9 N	.778 meters	0.280 meters	0.498 meters
0.6 kg / 5.88 N	.778 meters	0.080 meters	0.698 meters

Data Analysis

The table provided above contains some sample data whose format may be useful in your analysis. You may use the table to record your actual data and then use the data to construct a graph of applied force versus displacement. The data may be graphed manually or may be graphed using Excel. In either case you are required to make estimates of the accuracy of your result by making reasonable assumptions as to the accuracy of the quantities measured. A couple of points to ponder – the force you plot is mg – is a 100 gram mass going to exert a force of mg = .1kg x g = .1 x 9.8 Newtons = .98 N exactly?

Quantitatively determine the error associated with your determination of k. This will require that you estimate the error associated with the elongation of the spring and the error associated with the driving force, mg.

Part II

In this part of the experiment the period of the simple harmonic oscillator will be determined for various masses. Before getting into the details a little intuitive analysis is useful. If a mass is attached to the spring how do you believe the period will be affected by increasing the mass? By increasing the spring constant? It is always a good idea to have an intuitive idea of what you expect in a given situation. If the spring constant is increased one expects that a stiffer spring will apply a larger force and therefore increase the acceleration of the mass. If the acceleration is increased this will increase the velocity and thus one expects an increased spring constant will decrease the period. On the other hand if the mass is increased then one expects a longer period, as the acceleration will be decreased and thus the velocity and time to traverse a complete cycle will be increased. Based on this reasoning one guesses the period will decrease with increasing spring constant and increase with greater mass. So as a first guess perhaps the period is proportional to m/k. How might you verify that the period is proportional to m and inversely proportional to k?

The answer is to take some data and make a plot of T versus m for a fixed value of k[i]. (Note – see endnote) Imagine taking a spring and loading it with masses that are increased in say 100 gram increments from 100 to 2000 grams to obtain data. A set of such data is shown in figure 1 and a plot of the data is shown in figure 2.

Mass in Kg	Period in Sec	Mass in Kg	Period in Sec
0.10	0.77	1.10	2.13
0.20	0.99	1.20	2.22
0.30	1.18	1.30	2.30
0.40	1.33	1.40	2.39
0.50	1.47	1.50	2.47
0.60	1.60	1.60	2.55
0.70	1.72	1.70	2.63
0.80	1.83	1.80	2.70
0.90	1.94	1.90	2.77
1.00	2.04	2.00	2.84

Figure 1

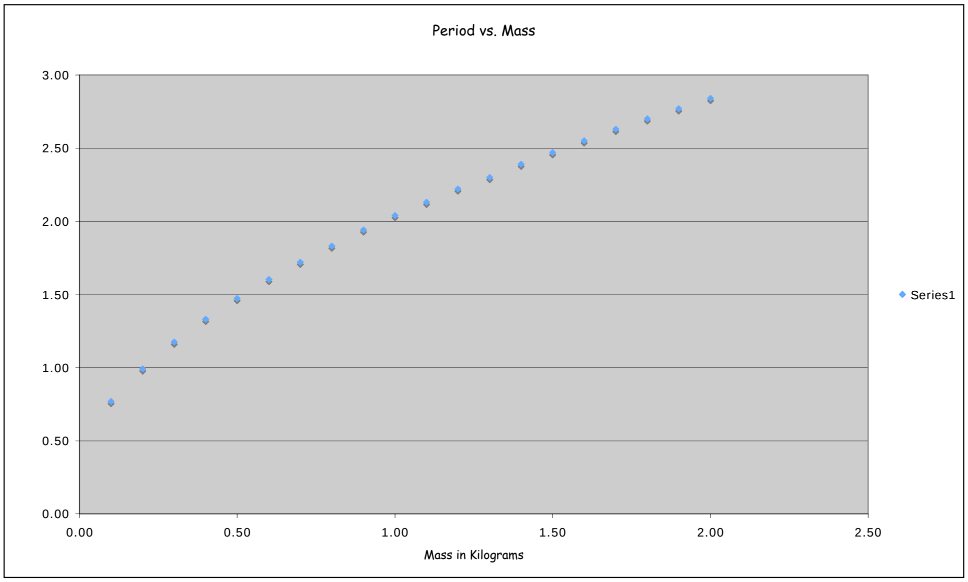

Figure 2

The graph suggests the period does not increase directly with the mass but has some more complicated dependence on the mass. If the period depends on mass raised to some power then a log-log plot of the data could be used to determine the dependence of T on m, that is assume T = constant x m^n , where n is an exponent. Taking the log of both sides of the equation gives: log T = log (constant) + n log (m), which is an equation of the form y = mx + b in which n is the slope. This could be done and is a very nice way to find such dependence (See Appendix II), however it is perhaps more instructive to calculate the dependence of T on both m and k.

In order to calculate the exact expression for the period we may use the fact that the total energy of the oscillator is conserved so that:

$$
\begin{aligned}
&E = \frac{1}{2}kx_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \\
&v^2 = \frac{k}{m}\left(x_0^2 - x^2\right) = \omega^2\left(x_0^2 - x^2\right) \text{ or } \; v = \omega\sqrt{\left(x_0^2 - x^2\right)}
\end{aligned}
\tag{1.1}
$$

But v = dx/dt so that (1.1) may be rearranged and integrated to find T

$$
\begin{aligned}
&\frac{dx}{dt} = \omega\sqrt{\left(x_0^2 - x^2\right)} \\
&\int_{x=0}^{x=x_0} \frac{dx}{\sqrt{\left(x_0^2 - x^2\right)}} = \int_0^{T/4} \omega dt
\end{aligned}
\tag{1.2}
$$

Unfortunately (1.2) is just a bit tricky to integrate. If you let x = $x_0$sint then dx = $x_0$sintdt and the limits of integration must be adjusted. If the lower limit on x = 0 then t = 0 will satisfy the transformation, and when x = x_0 the value of t = π /2 sp that:

$$
\begin{aligned}
&\int_{x=0}^{x=x_0} \frac{dx}{\sqrt{\left(x_0^2 - x^2\right)}} = \int_{t=0}^{t=\pi/2} \frac{x_0 \cos t dt}{\sqrt{\left(x_0^2 - x_0^2 \sin^2 t\right)}} = \int_{t=0}^{t=\pi/2} dt = \int_0^{T/4} \omega dt \\
&\int_{t=0}^{t=\pi/2} dt = \frac{\pi}{2} = \int_0^{T/4} \omega dt = \omega\frac{T}{4}
\end{aligned}
\tag{1.3}
$$

$$
T = 2\pi\sqrt{\frac{m}{k}}
$$

Equation shows that the period depends on the square root of m divided by k, so that for a fixed k the period squared is proportional to m. If you can follow the mathematics that is great but it is not a critical matter; rather the derivation is included as a matter of completeness for those interested in the details. The expression for the period is what is important for this lab.

Equation (1.3) assumes that the spring is massless and as this is not the case some allowance for the spring's mass should be included in (1.3), which may be accomplished by assuming the mass of the oscillator is m + m_{eff} where the additional term is for the effective mass of the spring. The effective mass of the spring will be less than the actual mass of the spring, a detail which may be understood by noting that at any instant when the mass is moving with a velocity v the spring will be moving at various speeds; the end connected to the moving mass will be moving at the same velocity as the mass whereas the end secured to the wall or support will be stationary. Thus at any instant the spring will have different speeds at different points along it length ranging from 0 to v. Equation (1.3) may thus be modified as:

$$
\begin{gathered}
T = 2\pi\sqrt{\frac{m + m_{eff}}{k}} \\
T^2 = \frac{4\pi^2}{k}m + \frac{4\pi^2}{k}m_{eff}
\end{gathered}
\tag{1.4}
$$

Data Collection

In the first part of the experiment the value of k was determined. In this part of the experiment it is desired to see if (1.4) accurately models the behavior of an actual oscillator. Using a mass of 150 grams start the oscillator in motion and measure the time required for it to complete 20 complete cycles. Do this three times for a total of 60 cycles. Increase the mass in increments of 50 grams until you have at least 6 data points. The table below may be used as guide for your data collection.

Mass	Trial 1	Trial 2	Trial 3	Period
150 grams	22.4 seconds	22.6 seconds	22.1 seconds	T = 1.12 seconds
150 grams				
200 grams				
300 grams				
350 grams				
400 grams				
450 grams				
500 grams				
550 grams				

In the table the time for 20 cycles is recorded under the corresponding trial number. The period is then found by summing the times for three trials and dividing the result by 60, which in the case of the sample data gives a period of 1.12 seconds. If you have an obvious outlier, discard this value and repeat the trial.

Data Analysis

Using the data obtained make a plot of T squared versus the mass. Plot the period squared on the vertical axis and the oscillator mass on the horizontal axis. Compare equation (1.4) with the equation for a straight line:

$$T^2 = \frac{4\pi^2}{k}m + \frac{4\pi^2}{k}m_{eff}$$

(1.5)

$$y = \eta x + b,\ \textit{where } \eta \textit{ is the slope.}$$

The period squared is associated with y, the slope of the straight line, η, is associated with 4π2/k, and the intercept is associated with (4π2/k)m_{eff}. Using Excel plot the data and determine the slope and intercept of the line. Equating the intercept with the expression for the effective mass find the value of the effective mass. Weigh the spring and find the ratio of the spring mass to the effective mass as determined by your analysis.

If the effective mass is ignored compare the values of the period, which are predicted by the formula for the period (equation 1.3) to those obtained experimentally and plot them on the same graph as the values experimentally obtained.

Exercises

Using the data in figure 1 duplicate the plot in figure 2 and plot the period squared as a function of m. Note the second graph is clearly a straight line. Find the slope and intercept of this line. What is the significance of the slope? The intercept?

Appendix I
Effective Mass

How can the effective mass of the spring be taken into account? If we assume the velocity of the spring is a linear function of position we may then calculate the kinetic energy of the spring at any instant by finding the kinetic energy of each element of the spring. (A problem in integration or fancy addition.) The figure below illustrates the mass moving to the right with a velocity v. The point at which the spring is attached to the mass is moving with velocity v, whereas the point attached to the wall is stationary. The midpoint of the spring is moving at velocity v/2.

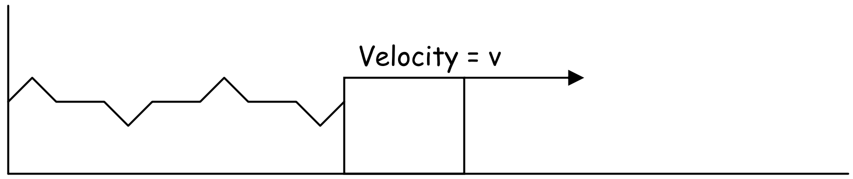

Figure 3

Assume the mass of the oscillator, in the figure, is at x = x_0, the spring is attached to the wall at x = 0, and that it is moving to the right with a velocity v. Since the velocity is assumed to be a linear function of x the expression v(x) = (x/x_0)v gives just what we need. When x = 0, v(x) = 0 and when x = x_0 v(x_0) = v. Try some other values if you are not convinced. If λ is the mass per unit length then dm = λdx and the element of the spring located at x has "hunk" of kinetic energy equal to:

$$dKE(x) = \frac{1}{2}v^2(x)\lambda dx \tag{1.6}$$

Where dKE(x) is the differential element of the kinetic energy for the element of mass λdx at v(x). If all the elements of the kinetic energy are added together, which is accomplished by integrating expression (1.6), the kinetic energy of the spring is obtained.

$$\begin{aligned}
&dKE(x) = \frac{1}{2}v^2(x)\lambda dx \\
&KE(x) = \frac{1}{2}\lambda\int_0^{x_0}\left(\frac{x}{x_0}\right)^2 v^2 dx = \frac{1}{2}\frac{\lambda v^2}{x_0^2}\int_0^{x_0} x^2 dx \\
&KE(x) = \frac{\lambda v^2}{2x_0^2}\left(\frac{x^3}{3}\right)\Bigg|_{x=0}^{x=x_0} = \frac{\lambda}{2x_0^2}\frac{x_0^3}{3}v^2 = \frac{\lambda}{2}\frac{x_0}{3}v^2 = \frac{1}{2}\frac{m}{3}v^2 = \frac{1}{2}m_{eff}v^2 \\
&where\ \lambda x_0 = m_{spring}
\end{aligned} \tag{1.7}$$

Hence the effective mass of the spring is given as one third the mass of the spring. Next if the kinetic energy of the system is calculated using this result one has:

$$KE_{oscillator} = \frac{1}{2}\left(m + \frac{m_{spring}}{3}\right)v^2 = \frac{1}{2}\left(m + m_{eff}\right)^2 \tag{1.8}$$

Finally this expression replaces the expression for the kinetic energy of the oscillator in equation (1.1) and the only thing that changes for the expression for the period is that m is replaced by m + m_{eff}.

Appendix II
Using a Log-Log Fit

As was mentioned above when data is obtained that is not linear there are more powerful methods to extract the relationship between experimentally determined variables. Examination of figure 2 shows that a plot of period versus mass is clearly not linear, however if we make the assumption that the period is proportional to the mass raised to some power then we may write:

$$T = \beta m^{n} \quad or \quad Ln(T) = Ln(\beta) + nLn(m) = nLn(m) + Ln(\beta) \tag{1.9}$$

If equation (1.9) is compared to the graph of a straight line one has:

$$\begin{gathered} Ln(T) = nLn(m) + Ln(\beta) \\ y = mx + b \ \ where: \\ y = Ln(T); \ \ m = n; \ \ x = Ln(m); \ \ and \ \ b = Ln(\beta) \end{gathered} \tag{1.10}$$

If this data is now plotted the slope of the straight line corresponds to the power to which m is to be raised and b, the intercept may be used to find the constant of proportionality. To illustrate consider the data in figure 4.

Mass	Period	Period Squared	Ln M	Ln T
1.00	1.99	3.95	0.00	0.69
2.00	2.81	7.90	0.69	1.03
3.00	3.44	11.84	1.10	1.24
4.00	3.97	15.79	1.39	1.38
5.00	4.44	19.74	1.61	1.49
6.00	4.87	23.69	1.79	1.58
7.00	5.26	27.63	1.95	1.66
8.00	5.62	31.58	2.08	1.73
9.00	5.96	35.53	2.20	1.79
10.00	6.28	39.48	2.30	1.84
11.00	6.59	43.43	2.40	1.89
12.00	6.88	47.37	2.48	1.93
13.00	7.16	51.32	2.56	1.97
14.00	7.43	55.27	2.64	2.01
15.00	7.70	59.22	2.71	2.04
16.00	7.95	63.17	2.77	2.07
17.00	8.19	67.11	2.83	2.10
18.00	8.43	71.06	2.89	2.13
19.00	8.66	75.01	2.94	2.16
20.00	8.89	78.96	3.00	2.18

Figure 4

For this sample data a plot of period versus mass give a graph, which is clearly not linear, and if you are familiar with the plots of various functions you may well be able to guess the exponent. But let us use this data to illustrate the usefulness of a log log plot, which can be used is situations when the answer is not as clear as in this case. Using the data from the table a plot of T versus m appears as shown in figure 5.

In figure 6 the plot of Ln(T) verses Ln(m) is shown. As can be seen from figure 6 the slope is .5, which means the exponent in equation (1.9) is ½ and thus the period is proportional to the square root of m. In addition the intercept is equal to .6866 from which the constant beta may be found as:

$$\beta = e^{b} = 1.986 = \frac{2\pi}{\sqrt{10}} \tag{1.11}$$

Beta is equal to the coefficient that was chosen to produce the data. If you wish try this method on the data in figure 1. The results will not be quite so perfect.

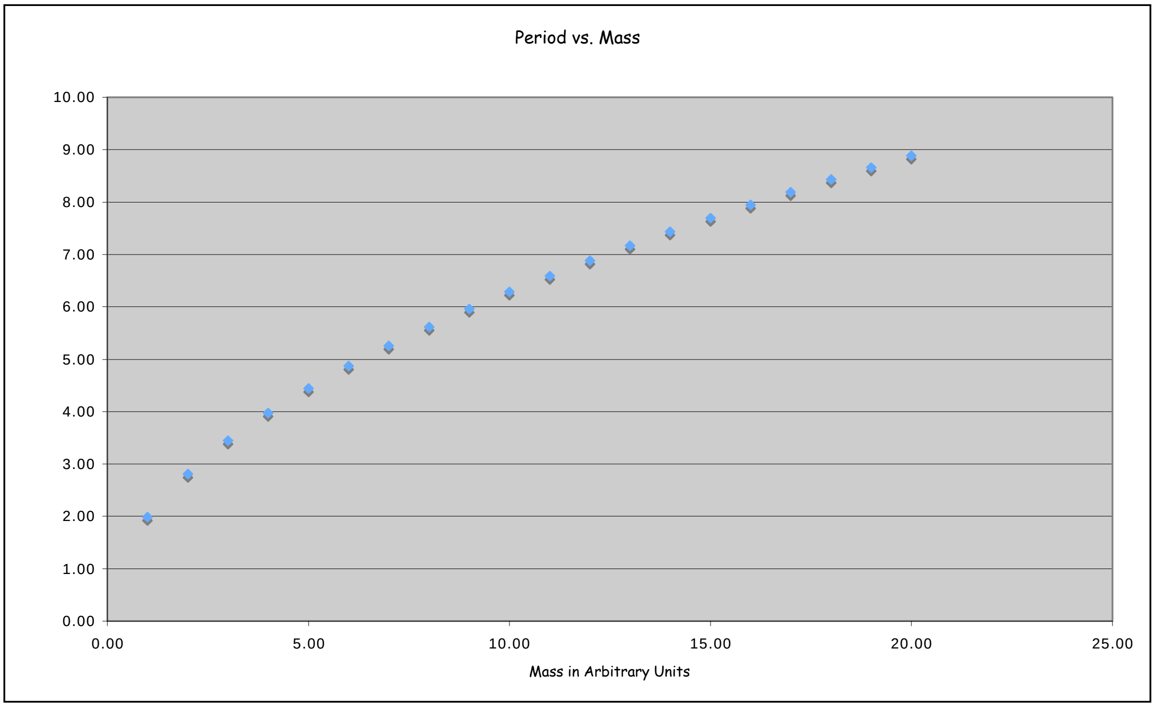

Figure 5

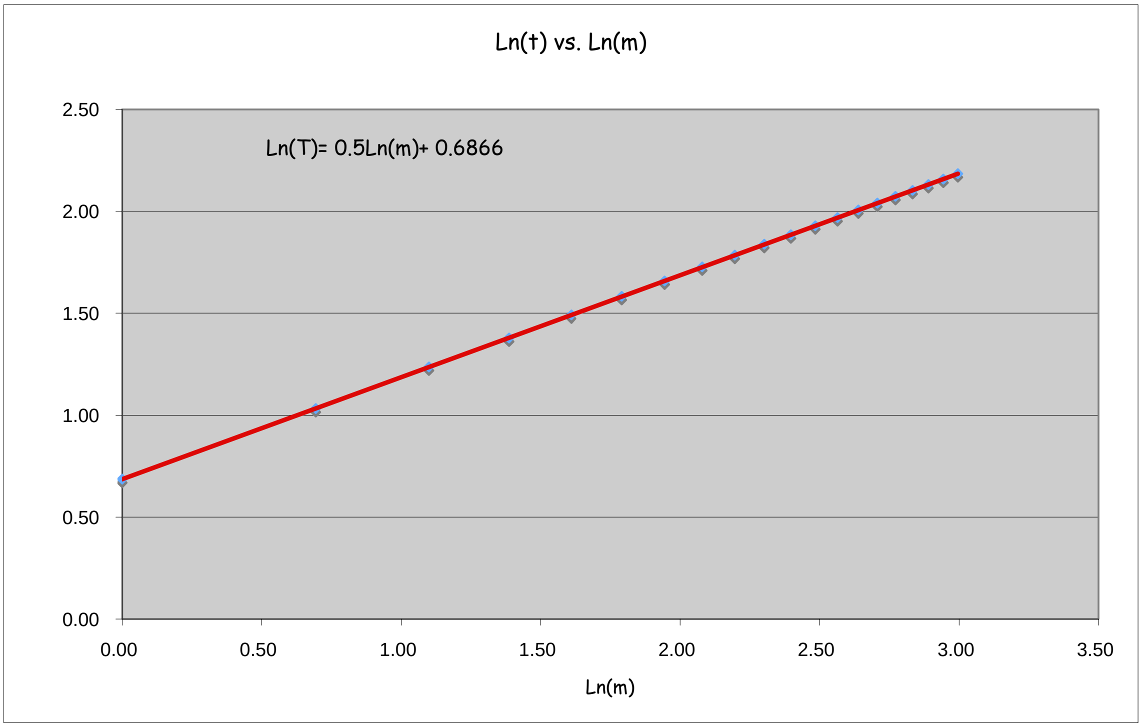

Figure 6

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[i] Of course you could keep the mass constant and change the spring. Do you see why this is not a good way to go about the experiment?