Chapter 1

The Real Number System

1.1 Sets and Operations on Sets

1. Let $A = \{-1, 0, 1, 2\}, B = \{-2, 3\}$, and $C = \{-2, 0, 1, 5\}$.

   a.
   - $A \cup B = \{-2, -1, 0, 1, 2, 3\}$
   - $B \cup C = \{-2, 0, 1, 3, 5\}$
   - $A \cap B = \phi$
   - $B \cap C = \{-2\}$
   - $A \cap C = \{0, 1\}$
   - $B \cup C = \{-2, 0, 1, 3, 5\}, A \cap (B \cup C) = \{0, 1\}$
   - $A \setminus B = \{-1, 0, 1, 2\}$
   - $C \setminus B = \{0, 1, 5\}$
   - $B \cup C = \{-2, 0, 1, 3, 5\}, A \setminus (B \cup C) = \{-1, 2\}$

3. a. This problem was solved as an example in class.

4. b. Let $x \in A \cup (B \cup C)$, then $x \in A$ or $x \in B \cup C$, so $x \in A$ or $x \in B$ or $x \in C$. Therefore $x \in A \cup B$ or $x \in C$, so $x \in (A \cup B) \cup C$ and $A \cup (B \cup C) \subset (A \cup B) \cup C$. A similar argument shows that $(A \cup B) \cup C \subset A \cup (B \cup C)$.

5. If $A \subset B$, prove that

   a. $A \cap B = A$

   - Part 1: We show that, for every choice of the sets A and B, $A \cap B \subset A$.
     Let $x \in A \cap B \Rightarrow x \in A$, so $A \cap B \subset A$.

- Part 2: We show that, if $A \subset B$, then $A \subset A \cap B$.
  Let $x \in A$. Then, since $A \subset B$, also $x \in B$. Hence $x \in A \cap B$, so $A \subset A \cap B$.

From Part 1 and Part 2, we may conclude that $A \cap B = A$.

b. $A \cup B = B$

- Part 1: We show that, if $A \subset B$, then $A \cup B \subset B$.
  Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Observe that, since $A \subset B$, $x \in A$ implies that $x \in B$. Therefore in either case $x \in B$, so $A \cup B \subset B$.
- Part 2: We show that, for every choice of the sets A and B, $B \subset A \cup B$.
  Let $x \in B$. Then $x \in A \cup B$, so $B \subset A \cup B$.

From Part 1 and Part 2, we may conclude that $A \cup B = B$.

6. Let $A \subset X$ and $A^c = X \setminus A$, show that

a. $A \cup A^c = X$.

- Part 1: We show that $A \cup A^c \subset X$.
  Let $x \in A \cup A^c \Rightarrow x \in A$ or $x \in A^c \Rightarrow x \in A$ or $x \in X \setminus A$. Since both A and $X \setminus A$ are subsets of X, this implies that $x \in X$, so $A \cup A^c \subset X$.
- Part 2: We show that $X \subset A \cup A^c$.
  Let $x \in X \Rightarrow x \in A$ or $x \in X \setminus A \Rightarrow x \in A$ or $x \in A^c \Rightarrow x \in A \cup A^c$, so $X \subset A \cup A^c$.

From Part 1 and Part 2, we conclude that $X = A \cup A^c$.

b. $A \cap A^c = \phi$.

This is most easily proved by showing that $A \cap A^c$ is a set with no elements. We use a proof by contradiction. Suppose that $A \cap A^c$ contains an element x, then $x \in A$ and $x \in A^c$. This means that $x \in A$ and $x \notin A$, a contradiction. We conclude that $A \cap A^c = \phi$.

c. $(A^c)^c = A$.

- Part 1: We show that $(A^c)^c \subset A$.
  Let $x \in (A^c)^c \Rightarrow x \notin A^c \Rightarrow x \in A$, so $(A^c)^c \subset A$.
- Part 2: We show that $A \subset (A^c)^c$.
  Let $x \in A \Rightarrow x \notin A^c \Rightarrow x \in (A^c)^c$, so $A \subset (A^c)^c$.

From Part 1 and Part 2, we conclude that $(A^c)^c = A$.

10.
- To show that $A \cap B$ and $A \setminus B$ are disjoint, we use a proof by contradiction. Suppose $x \in A \cap B$ and $x \in A \setminus B$, then $x \in B$ and $x \notin B$, a contradiction. We conclude that $A \cap B$ and $A \setminus B$ are disjoint.
- Finally, we prove that $A = (A \cap B) \cup (A \setminus B)$.

– Part 1: We show that $A \subset (A \cap B) \cup (A \setminus B)$.
Let $x \in A \Rightarrow (x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \notin B) \Rightarrow x \in A \cap B$ or $x \in A \setminus B \Rightarrow x \in (A \cap B) \cup (A \setminus B)$, so $A \subset (A \cap B) \cup (A \setminus B)$.
– Part 2: We show that $(A \cap B) \cup (A \setminus B) \subset A$.
Let $x \in (A \cap B) \cup (A \setminus B) \Rightarrow x \in A \cap B$ or $x \in A \setminus B$. In either case $x \in A$, so $(A \cap B) \cup (A \setminus B) \subset A$.

From Part 1 and Part 2, we conclude that $A = (A \cap B) \cup (A \setminus B)$.

1.2 Functions

1. a. No, f is not a function from A to B, because not every element of A occurs as the first component of an element of f. In particular $1 \in A$, but 1 is not the first component of any of the elements of f.

   b. No, g is not a function from A to B, because not every element of A occurs as the first component of precisely one element of g. For instance $(1, -1)$ and $(1, 3)$ both are elements of g. Moreover, $-1 \notin \mathbb{N} = B$, this provides a reason to conclude that g cannot be a function into B.

   c. No, h is not a function from A to $B = \mathbb{N}$, because $(2, -1) \in h$, so h is not a subset of $A \times B$.

   d. Yes, k is a function from A to B, because k is a subset of $A \times B$ and every element of A occurs as the first component of precisely one element of k.

2. a. See the "Hints and Solutions to Selected Exercises" section of the textbook.

   b. Yes, if $C = \{x : -1 \leq x \leq 1\}$ and $D = \{x : y \geq 0\}$, then B is a function from C to D, because B is a subset of $C \times D$ and every element of C occurs as the first component of precisely one element of B. Please note that, as is often the case, the choice of D is not unique. We could for instance have chosen $D = \mathbb{R}$, or $D = \{y : 0 \leq y \leq 1\}$.

3. a. See the "Hints and Solutions to Selected Exercises" section of the textbook.

   b.
   - $f(E) = \{f(n) : n \in E\} = \{f(1), f(3), f(5), f(7)\} = \{1, 5, 9, 13\}$
   - $f^{-1}(E) = \{n \in \mathbb{N} : f(n) \in E\} = \{1, 2, 3, 4\}$

   c.
   - $f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\} = \{2n - 1 : n \in \mathbb{N}\} = \{1, 3, 5, \cdots\}$
   - $f^{-1}(\mathbb{N}) = \{n \in \mathbb{N} : f(n) \in \mathbb{N}\} = \mathbb{N}$

4. a. See the "Hints and Solutions to Selected Exercises" section of the textbook.

   b.
   - Let $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Then, since $f(x) = x^3 + 1$ is increasing on $\mathbb{R}$, $f(x_1) < f(x_2)$. Hence f is one-to-one.

- To show that f is onto, we let $y \in \mathbb{R}$ and show that $\exists x \in \mathbb{R}$ with $x^3 + 1 = y$. Solve this equation for x^3 and find $x^3 = y - 1$. Finally, because $g : \mathbb{R} \to \mathbb{R}, g(x) = \sqrt[3]{x}$ is single valued, we may take the cube root of both sides of the equation $x^3 = y - 1$ and find $x = \sqrt[3]{y-1}$. This proves that f is onto.

c. See the "Hints and Solutions to Selected Exercises" section of the textbook.

6. b. $f(x) = 3x - 2, \mathrm{Dom}\, f = \mathbb{R}$.

- $\mathrm{Range}\, f = \{f(x) : x \in \mathrm{Dom}\, f\} = \mathbb{R}$.
- To determine if f is one-to-one, we investigate if $(x_1, y) \in f$ and $(x_2, y) \in f$ implies that $x_1 = x_2$. Suppose $y = f(x_1)$ and $y = f(x_2)$, then $f(x_1) = f(x_2)$, so

$$5x_1 + 4 = 5x_2 + 4$$

Clearly, this implies $x_1 = x_2$. Hence f is one-to-one.
- To find f^{-1} we let $y \in \mathrm{Range}\, f = \mathbb{R}$ and solve the equation $y = f(x) = 5x + 4$ for x. Elementary arithmetic shows that $x = \frac{1}{5}(y-4)$ Therefore $f^{-1}(y) = \frac{1}{5}(y-4), \mathrm{Dom}\, f^{-1} = \mathrm{Range}\, f = \mathbb{R}$.

d. $f(x) = \frac{x}{1-x}, \mathrm{Dom}\, f = \{x \in \mathbb{R} : 0 < x < 1\}$.

- $\mathrm{Range}\, f = \{f(x) : x \in \mathrm{Dom}\, f\} = \{y : y > 0\}$
- To determine if f is one-to-one, we investigate if $(x_1, y) \in f$ and $(x_2, y) \in f$ implies that $x_1 = x_2$. Suppose $y = f(x_1)$ and $y = f(x_2)$, then $f(x_1) = f(x_2)$, so

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

This implies $x_1(1-x_2) = x_2(1-x_1)$ and therefore $x_1 = x_2$. Hence f is one-to-one.
- To find f^{-1} we let $y \in \mathrm{Range}\, f = \{y : y > 0\}$ and solve the equation $y = f(x) = \frac{x}{1-x}$ for x. Observe that $y(1-x) = x$, so $x + yx = y$ and $x = \frac{y}{1+y}$. Therefore $f^{-1}(y) = \frac{y}{1+y}, \mathrm{Dom}\, f^{-1} = \mathrm{Range}\, f = \{y : y > 0\}$.

e. $f(x) = \frac{1}{x^2+1}, \mathrm{Dom}\, f = \{x \in \mathbb{R} : -1 \le x \le 1\}$.

- $\mathrm{Range}\, f = \{f(x) : x \in \mathrm{Dom}\, f\} = \left\{y : \frac{1}{2} \le y \le 1\right\}$
- Observe that $f(-1) = f(1) = \frac{1}{2}$, so f is not one-to-one.

8. c. If f is a function from A to B, and $B_1 \subset B$, then show that $f^{-1}(B \setminus B_1) = A \setminus f^{-1}(B_1)$.

- Part 1: We show that $f^{-1}(B \setminus B_1) \subset A \setminus f^{-1}(B_1)$.
Let $x \in f^{-1}(B \setminus B_1)$, then $y = f(x) \in B \setminus B_1$, so $y = f(x) \notin B_1$, thus $x \notin f^{-1}(B_1)$. Moreover, since $f^{-1}(B \setminus B_1) \subset A$, $x \in A$. Therefore $x \in A \setminus f^{-1}(B_1)$. We conclude that $f^{-1}(B \setminus B_1) \subset A \setminus f^{-1}(B_1)$.

- Part 2: We show that $A \setminus f^{-1}(B_1) \subset f^{-1}(B \setminus B_1)$.
  Let $x \in A \setminus f^{-1}(B_1)$, then $x \notin f^{-1}(B_1)$, so $y = f(x) \notin B_1$. Moreover, since f is a function from A to B, $y = f(x) \in B$. Therefore $y = f(x) \in B \setminus B_1$ and $x \in f^{-1}(B \setminus B_1)$. We conclude that $A \setminus f^{-1}(B_1) \subset f^{-1}(B \setminus B_1)$.

From Part 1 and Part 2, it follows that $f^{-1}(B \setminus B_1) = A \setminus f^{-1}(B_1)$.

9. Let $f : A \to B$ and let $F \subset A$.

   a. Show that $f(A) \setminus f(F) \subset f(A \setminus F)$.
   Let $y \in f(A) \setminus f(F)$, then $y \in f(A)$ and $y \notin f(F)$, so there exists an $x \in A$ such that $y = f(x)$ and, since $y = f(x) \notin f(F)$, $x \notin F$. Therefore $x \in A \setminus F$, and $y = f(x) \in f(A \setminus F)$. This completes the proof.

   b. Take for instance $A = \{x : -3 \le x \le 3\}$, $f : A \to \mathbb{R}$, $f(x) = x^2$, and $F = \{x : -1 \le x \le 0\}$. Then

   - $f(A) = \{y : 0 \le y \le 9\}$
   - $f(F) = \{y : 0 \le y \le 1\}$
   - $f(A) \setminus f(F) = \{y : 1 < y \le 9\}$
   - $A \setminus F = \{x : -3 \le x < -1 \text{ or } 0 < x \le 3\}$, and
   - $f(A \setminus F) = \{y : 0 < y \le 9\} \ne \{y : 1 < y \le 9\} = f(A) \setminus f(F)$

BE. **Blackboard Exercise, August 31.**

Show that the function $f : [0,1] \to \mathbb{R}$, $f(x) = \frac{2x}{1+3x}$ is one-to-one.

Solution

Let $x_1, x_2 \in [0,1]$, and let both (x_1, y) and (x_2, y) be elements of f. That is $f(x_1) = f(x_2)$. Hence

$$\frac{2x_1}{1+3x_1} = \frac{2x_2}{1+3x_2}$$

and

$$2x_1(1+3x_2) = 2x_2(1+3x_1)$$

Expanding the products yields

$$2x_1 + 6x_1x_2 = 2x_2 + 6x_2x_1$$

so $x_1 = x_2$, and we may conclude that f is one-to-one.

1.3 Mathematical Induction

1. In these solutions it is understood that the variables n and k are natural numbers.

   a. For $n = 1$, both $1 + 2 + \cdots + n$ and $\frac{n(n+1)}{2}$ reduce to 1, so formula

   $$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

   is true for $n = 1$. Next assume that it is true for $n = k \geq 1$, and observe that under these circumstances

   $$\begin{aligned}(1 + 2 + \cdots + k) + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ = \left(\frac{k}{2} + 1\right)(k+1) &= \frac{1}{2}(k+2)(k+1) = \frac{1}{2}(k+1)((k+1)+1)\end{aligned}$$

   Clearly this means that the formula is true for $n = k + 1$, and by the principle of mathematical induction we may conclude that the formula is true for all $n \in \mathbb{N}$.

   e. For $n = 1$, both $2 + 2^2 + \cdots + 2^n$ and $2(2^n - 1)$ reduce to 2, so formula

   $$2 + 2^2 + \cdots + 2^n = 2(2^n - 1)$$

   is true for $n = 1$. Next assume that it is true for $n = k \geq 1$, and observe that under these circumstances

   $$\left(2 + 2^2 + \cdots + 2^k\right) + 2^{k+1} = 2\left(2^k - 1\right) + 2^{k+1} = 2\left(2^k - 1\right) + 2\left(2^k\right) = 2\left(2^{k+1} - 1\right)$$

   Clearly this means that the formula is true for $n = k + 1$, and by the principle of mathematical induction we may conclude that the formula is true for all $n \in \mathbb{N}$.

2. In these solutions it is understood that the variables n and k are natural numbers.

   b. Observe $2^5 = 32$, while $5^2 = 25$, so the statement $2^n > n^2$ is true for $n = 5$. Next we assume that $2^k > k^2$ for some $k \geq 5$ and prove that this information implies that $2^{k+1} > (k+1)^2$. Clearly

   $$\begin{aligned}2^{k+1} &= 2 \cdot 2^k > 2k^2 = (k+1)^2 + \left\{2k^2 - (k+1)^2\right\} \\ &= (k+1)^2 + \left\{k^2 - 2k - 1\right\} \\ &= (k+1)^2 + \left(\frac{1}{2}k^2 - 2k\right) + \left(\frac{1}{2}k^2 - 1\right) \\ &= (k+1)^2 + \left(\frac{1}{2}k - 2\right)k + \left(\frac{1}{2}k^2 - 1\right)\end{aligned}$$

For $k \geq 5$ the expression $\left(\frac{1}{2}k - 2\right)k + \left(\frac{1}{2}k^2 - 1\right)$ is greater than zero, so $2^{k+1} > (k+1)^2$. By the principle of mathematical induction we may conclude that $2^n > n^2$ for all natural numbers $n \geq 5$.

c. See the "Hints and Solutions to Selected Exercises" section of the textbook

e. Observe that $\sum_{i=1}^{3} i^3 = 36$, while $\frac{1}{2}3^4 = \frac{81}{2}$, so the formula

$$\sum_{i=1}^{n} i^3 < \frac{1}{2}n^4$$

is true for $n = 3$. Next we assume that $\sum_{i=1}^{k} i^3 < \frac{1}{2}k^4$ is true for some $k \geq 3$ and prove that this information implies

$$\sum_{i=1}^{k+1} i^3 < \frac{1}{2}(k+1)^4$$

We start at the left side of this inequality and steadily work towards the desired result.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 < \frac{1}{2}k^4 + (k+1)^3 \\
&= \frac{1}{2}(k+1)^4 - \left[\frac{1}{2}(k+1)^4 - \frac{1}{2}k^4 - (k+1)^3\right]
\end{aligned}$$

Expanding the expression between square brackets yields

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &< \frac{1}{2}(k+1)^4 - \left[\frac{1}{2}(k+1)^4 - \frac{1}{2}k^4 - (k+1)^3\right] \\
&= \frac{1}{2}(k+1)^4 - \left[k^3 - k - \frac{1}{2}\right] \\
&= \frac{1}{2}(k+1)^4 - \left[\left(\frac{1}{2}k^3 - k\right) + \left(\frac{1}{2}k^3 - \frac{1}{2}\right)\right] \\
&= \frac{1}{2}(k+1)^4 - \left[\frac{1}{2}k\left(k^2 - 2\right) + \frac{1}{2}\left(k^3 - 1\right)\right] \\
&< \frac{1}{2}(k+1)^4
\end{aligned}$$

By the principle of mathematical induction we may conclude that $\sum_{i=1}^{n} i^3 < \frac{1}{2}n^4$ for all natural numbers $n \geq 3$.

1.4 The Least Upper Bound Property

1. b. Show that if $a \in \mathbb{R}$, then $(-1) \cdot a = -a$.

   We show that both sides are an additive inverse of a.

   - $a + (-a) = 0$, by definition.
   - $a + (-1) \cdot a = a \cdot 1 + a \cdot (-1) = a \cdot (1 + (-1)) = a \cdot 0$. During the class meeting of September 7, we proved that $a \cdot 0 = 0$ for all $a \in \mathbb{R}$. Hence $a + (-1) \cdot a = a \cdot 0 = 0$

   Also during the September 7 meeting, we showed that the additive inverse is unique, so $(-1) \cdot a = -a$.

2. a. If $a \in \mathbb{R}$, prove the following: If $a \neq 0$, then $\frac{1}{a} \neq 0$.

   Use a proof by contradiction. Assume $a \neq 0$, and $\frac{1}{a} = a^{-1} = 0$, then

   $$1 = a \cdot a^{-1} = a \cdot 0 = 0$$

   a contradiction, because by axiom 6, $1 \neq 0$. We conclude that $a^{-1} \neq 0$.

   b. If $a, b \in \mathbb{R}$, prove that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

   Assume that $a \neq 0$, then a^{-1} exists, so we can multiply both sides of the equality $a \cdot b = 0$ by a^{-1} and obtain

   $$a^{-1} \cdot a \cdot b = a^{-1} \cdot 0$$

   which implies that $b = 0$. Similarly, if $b \neq 0$, then $a = 0$. This completes the proof.

3. c. Show that if $a \in \mathbb{R}$, and $a \neq 0$, then $a^2 > 0$.

   Since $a \in \mathbb{R}$, and $a \neq 0$, either $a \in \mathbb{P}$, or $-a \in \mathbb{P}$.

   - If $a \in \mathbb{P}$, then $a^2 = a \cdot a \in \mathbb{P}$, so $a^2 > 0$.
   - If $-a \in \mathbb{P}$, then $(-a)^2 = (-a) \cdot (-a) \in \mathbb{P}$. In the class meeting of September 7, we proved that $\forall a, b \in \mathbb{R}$, $(-a) \cdot b = -(a \cdot b)$. Hence

     $$(-a) \cdot (-a) = -(a \cdot (-a)) = -((-a) \cdot a) = -(-(a \cdot a))$$

     In the same class meeting of September 7, we proved that $\forall a \in \mathbb{R}$, $-(-a) = a$, therefore

     $$(-a) \cdot (-a) = -(-(a \cdot a)) = a \cdot a = a^2$$

     Hence , $a^2 \in \mathbb{P}$, so again $a^2 > 0$.

   This completes the proof.

4. In Exercise 3c we proved that a square is never negative, so

$$0 \leq (a-b)^2 \leq a^2 - 2ab + b^2$$

Bring the ab term to the left and divide both sides of the inequality by 2

$$2ab \leq a^2 + b^2, \text{ and}$$
$$ab \leq \frac{1}{2}\left(a^2 + b^2\right)$$

5. Find the supremum and infimum of each of the following sets.

   b. $B = \left\{\sin n\frac{\pi}{2} : n \in \mathbb{N}\right\} = \{-1, 0, 1\}$. $\inf B = -1$ and $\sup B = 1$.

   c. $C = \{n \cos n\pi : n \in \mathbb{N}\} = \{(-1)^n n : n \in \mathbb{N}\}$. This set is neither bounded above, nor bounded below. We say that $\inf C = -\infty$ and $\sup C = \infty$.

   f. $F = \left\{(-1)^n - \frac{1}{n} : n \in \mathbb{N}\right\} = \left\{1 - \frac{1}{2m} : m \in \mathbb{N}\right\} \cup \left\{-1 - \frac{1}{2m-1} : m \in \mathbb{N}\right\}$. Observe that

   $$\inf F = \inf\left\{-1 - \frac{1}{2m-1} : m \in \mathbb{N}\right\} = -2$$

   and

   $$\sup F = \sup\left\{1 - \frac{1}{2m} : m \in \mathbb{N}\right\} = 1$$

   h. $H = \left\{\left(\frac{3}{2} + (-1)^n\right)^n : n \in \mathbb{N}\right\} = \left\{\left(\frac{5}{2}\right)^{2m} : m \in \mathbb{N}\right\} \cup \left\{\left(\frac{1}{2}\right)^{2m-1} : m \in \mathbb{N}\right\}$. So, $\inf H = \inf\left\{\left(\frac{1}{2}\right)^{2m-1} : m \in \mathbb{N}\right\} = 0$, and since the set is not bounded above, we write $\sup H = \infty$.

10. Let $A \subset \mathbb{R}, A \neq \phi$, and let $-A = \{-x : x \in A\}$. Prove that

$$\inf A = -\sup(-A)$$

To prove this, let $\alpha = \inf A$, and $\beta = -\sup(-A)$. If A is not bounded below, then $-A$ is not bounded above, and the result is trivial. If A is bounded below, then $-A$ is bounded above, and we will show that $\alpha \leq \beta$, and $\beta \leq \alpha$.

- Observe that $\alpha \leq x$ for all $x \in A$. So $-\alpha \geq -x$ for all $x \in A$. We conclude that $-\alpha$ is an upper bound of $-A$. Therefore $-\beta \leq -\alpha$, and thus $\alpha \leq \beta$.
- Similarly, $-x \leq -\beta$ for all $x \in A$. So $x \geq \beta$ for all $x \in A$. We conclude that β is a lower bound of A. Therefore $\beta \leq \alpha$.

Since $\alpha \leq \beta$, and $\beta \leq \alpha$, we have shown that $\alpha = \beta$. This completes the proof.

11. Use the least upper bound property of $\mathbb{R}$ to prove that every nonempty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

    Let A denote a nonempty subset of $\mathbb{R}$ that is bounded below. Then, the set

$$-A = \{-x : x \in A\}$$

    is a nonempty subset of $\mathbb{R}$ which is bounded above. By the least upper bound property of $\mathbb{R}$, $\sup(-A)$ exists in $\mathbb{R}$. Finally, using the result of Exercise 10, we may conclude that $\inf A = -\sup(-A)$.

12. If A and B are nonempty subsets of $\mathbb{R}$ with $A \subset B$, prove that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

    - Part 1: Since $A \subset B$, every lower bound of B is also a lower bound of A. Therefore, $\inf B$ is a lower bound of A and thus $\inf B \leq \inf A$.
    - Part 2: Since A is nonempty, there exists an $x \in A$, and by definition $\inf A \leq x \leq \sup A$.
    - Part 3: Since $A \subset B$, every upper bound of B is also an upper bound of A. Therefore, $\sup B$ is an upper bound of A and thus $\sup A \leq \sup B$.

15. Let f, g be real-valued functions defined on a nonempty set X satisfying Range f and Range g are bounded subsets of $\mathbb{R}$. Prove each of the following.

    b. $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}$.
    Let $\alpha = \inf\{f(x) : x \in X\}$ and $\beta = \inf\{g(x) : x \in X\}$. Then for all $x \in X$

$$\alpha + \beta \leq f(x) + g(x)$$

    Hence $\alpha + \beta$ is a lower bound of $\{f(x) + g(x) : x \in X\}$, so

$$\alpha + \beta \leq \inf\{f(x) + g(x) : x \in X\}$$

1.5 Consequences of the Least Upper Bound Property

3. Let $r \in \mathbb{Q}, r \neq 0$, and let x be irrational. Show that $r + x$ and rx are irrational.

   In both cases, we use a proof by contradiction.

   - Suppose $r + x = q_1 \in \mathbb{Q}$. Then $x = q_1 - r \in \mathbb{Q}$, a contradiction.
   - Suppose $rx = q_2 \in \mathbb{Q}$. Then $x = \frac{q_2}{r} \in \mathbb{Q}$, a contradiction.

6. a. Prove that between any two rational numbers, there exists an irrational number.

   The quickest way to prove this property is to use Theorem 1.5.2. Let $r_1, r_2 \in \mathbb{Q}$ with $r_1 < r_2$. Then $\sqrt{2}r_1 < \sqrt{2}r_2$, while both $\sqrt{2}r_1$ and $\sqrt{2}r_2$ are real numbers. According to Theorem 1.5.2, there exists an $r \in \mathbb{Q}$ with $\sqrt{2}r_1 < r < \sqrt{2}r_2$. Division of all three parts of this double inequality by $\sqrt{2}$, yields

$$r_1 < \frac{r}{2}\sqrt{2} < r_2$$

   By the result of Exercise 3, $\frac{r}{2}\sqrt{2}$ is irrational, so the proof is complete.

   There is a tendency in mathematics not to bring out the big guns unless we have to. This property can be proved without using Theorem 1.5.2., (the big gun). The following proof applies only elementary means, but it is not necessarily easier to conceive. It is based on the parametric representation of the line segment between the points with coordinates r_1 and r_2, on the real axis. Let

$$x : [0, 1] \to [r_1, r_2], x(t) = r_1 + t(r_2 - r_1)$$

   Recall that $\frac{\sqrt{2}}{2}$ is irrational and $\frac{\sqrt{2}}{2} \in (0, 1)$. Therefore, $r_1 < x\left(\frac{\sqrt{2}}{2}\right) < r_2$, and, by the result of Exercise 3, $x\left(\frac{\sqrt{2}}{2}\right)$ is irrational.

   b. Prove that between any two real numbers, there exists an irrational number.

   Here we need the big gun, Theorem 1.5.2. The proof is similar to the first proof given in Part a. Let $x, y \in \mathbb{R}$ with $x < y$. Then $\sqrt{2}x < \sqrt{2}y$, while both $\sqrt{2}x$ and $\sqrt{2}y$ are real numbers. According to Theorem 1.5.2, there exists an $r \in \mathbb{Q}$ with $\sqrt{2}x < r < \sqrt{2}y$. Division of all three parts of this double inequality by $\sqrt{2}$, yields

$$x < \frac{r}{2}\sqrt{2} < y$$

   By the result of Exercise 3, $\frac{r}{2}\sqrt{2}$ is irrational, so the proof is complete.

7. If $x > 0$, show that there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < x$.

   To prove this statement we recall that according to the Archimedean property of $\mathbb{N}$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < x$. To complete the proof, we remark that for all $m \in \mathbb{N}, 2^m > m$, so

$$\frac{1}{2^n} < \frac{1}{n} < x$$

   Note that the inequality $2^m > m$ is easily proved by induction.

   - $2^1 = 2 > 1$, therefore the inequality is true for $m = 1$.
   - Suppose $k \geq 1$ and $2^k > k$, then

$$2^{k+1} = 2 \cdot 2^k > 2k = k + 1 + [2k - (k + 1)] = k + 1 + (k - 1) \geq k + 1$$

   By the principle of mathematical induction, the inequality $2^m > m$ holds for all $m \in \mathbb{N}$.

1.6 Binary and Ternary Expansion

1.7 Countable and Uncountable Sets

2. Let $\mathbb{O}$ denote the set of odd positive integers. Prove that $\mathbb{O} \sim \mathbb{N}$.

   Observe that $f : \mathbb{N} \to \mathbb{O}$, $f(n) = 2n - 1$ is one-to-one and onto.

4. a. If $a, b \in R$ with $a < b$, prove that $(a, b) \sim (0, 1)$.
   Observe that $f : (0,1) \to (a,b)\,,\, f(x) = a + x(b-a)$ is one-to-one and onto.

   b. Prove that $(0, 1) \sim (0, \infty)$.
   For instance, the function

$$f : (0,1) \to (0,\infty)\,,\, f(x) = \frac{x}{1-x}$$

   is one-to-one and onto. So $(0, 1) \sim (0, \infty)$.
   An alternative example of a one-to-one mapping of $(0, 1)$ onto $(0, \infty)$ is given by

$$g : (0,1) \to (0,\infty)\,,\, g(x) = \tan\frac{\pi x}{2}$$

8. Find $\cup_{n=1}^{\infty} A_n$ and $\cap_{n=1}^{\infty} A_n$ for each of the following sequences of sets $\{A_n\}$.

   a. $A_n = \{x \in \mathbb{R} : -n < x < n\}\,, n \in \mathbb{N}$
      - $\cup_{n=1}^{\infty} A_n = \mathbb{R}$
      - $\cap_{n=1}^{\infty} A_n = \{x \in \mathbb{R} : -1 < x < 1\} = (-1, 1)$

   b. $A_n = \left\{x \in \mathbb{R} : -\frac{1}{n} < x < 1\right\}, n \in \mathbb{N}$
      - $\cup_{n=1}^{\infty} A_n = (-1, 1)$
      - $\cap_{n=1}^{\infty} A_n = [0, 1)$

   e. $A_n = \left\{x \in \mathbb{R} : \frac{1}{n} \le x \le 1 - \frac{1}{n}\right\}, n \in \mathbb{N}, n \ge 2$
      - $\cup_{n=1}^{\infty} A_n = (0, 1)$
      - $\cap_{n=1}^{\infty} A_n = \left\{\frac{1}{2}\right\}$

   f. $A_n = \{x \in \mathbb{R} : n \le x < \infty\}\,, n \in \mathbb{N}$
      - $\cup_{n=1}^{\infty} A_n = [1, \infty)$
      - $\cap_{n=1}^{\infty} A_n = \phi$

12. Let f be a function from X into Y.

b. If $\{B_\alpha\}_{\alpha\in A}$ is a family of subsets of Y, prove that $f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right) = \cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)$.

Use the common approach. First prove that $f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)$ is a subset of $\cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)$, then show that $\cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)$ is a subset of $f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)$.

- Since $\cap_{\alpha\in A}B_\alpha \subset B_\alpha$ for all $\alpha\in A$, $f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)\subset f^{-1}\left(B_\alpha\right)$ for all $\alpha\in A$. Therefore, $f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)\subset \cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)$.
- Let $x\in\cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)$. Then $x\in f^{-1}\left(B_\alpha\right)$ for all $\alpha\in A$, so $f\left(x\right)\in B_\alpha$ for all $\alpha\in A$. Hence, $f\left(x\right)\in\cap_\alpha\left(B_\alpha\right)$. This implies that $x\in f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)$. We conclude that $\cap_{\alpha\in A}f^{-1}\left(B_\alpha\right)\subset f^{-1}\left(\cap_{\alpha\in A}B_\alpha\right)$.

This completes the proof.

13. a. If A is a countable subset of an uncountable set X, prove that $X\setminus A$ is uncountable.

We use a proof by contradiction. Suppose $X\setminus A$ is at most countable. Then,

- if $X\setminus A$ is finite, $X = A\cup(X\setminus A)$ is surely countable, a contradiction
- if $X\setminus A$ is countable, then by Theorem 1.7.15, $X = A\cup(X\setminus A)$ is once again countable, a contradiction

This completes the proof.

Remark: Theorem 1.7.15 implies that the union of two countable sets A and B is countable. To see this, choose $E_1 = A$ and $E_n = B$ for all $n\in\mathbb{N}$ with $n\geq 2$.

b. Prove that the set of irrational numbers is uncountable.

Apply the result of Part a with $X = \mathbb{R}$ and $A = \mathbb{Q}$.

14. Suppose f is a function from X into Y. If the range of f is uncountable, prove that X is uncountable.

Clearly, X cannot be finite. Suppose that X is countable. We will use this assumption to create a function from $\mathbb{N}$ onto the range of f. Then, by Theorem 1.7.7, the range of f is at most countable, which provides the contradiction we need.

If X is countable then there exists a function $g:\mathbb{N}\to X$ that is one-to-one and onto. We do not care about the one-to-one, but we need the onto. Observe that $s = f\circ g$ is a function from $\mathbb{N}$ onto the range of f. This completes the proof.