

Chapter 2

Sequences of Real Numbers

2.1 Convergent Sequences

1. c. Prove that $|x| = \sqrt{x^2}$ for all $x \in \mathbb{R}$.

This proof involves both the definition of the absolute value and the definition of the radical.

Recall that if $a \geq 0$, then \sqrt{a} equals the nonnegative number b , such that $b^2 = a$.

- Part 1: $x > 0$

In this case, $|x| = x$ and $\sqrt{x^2} = x$, so $|x| = \sqrt{x^2}$.

- Part 2: $x \leq 0$

In this case, $|x| = -x$ and $\sqrt{x^2} = -x$, and again $|x| = \sqrt{x^2}$.

- e. Prove that $-|x| \leq x \leq |x|$ for all $x \in \mathbb{R}$.

From the definition of the absolute value it follows that either $x = -|x|$ or $x = |x|$. Moreover, since $|x|$ is nonnegative,

$$-|x| \leq |x|$$

This implies the desired result, $-|x| \leq x \leq |x|$.

6. Determine all $x \in \mathbb{R}$ that satisfy each of the following inequalities.

- a. $|3x - 2| \leq 11$

This inequality is equivalent to $-11 \leq 3x - 2 \leq 11$, so $-9 \leq 3x \leq 13$. Thus

$$-3 \leq x \leq \frac{13}{3}$$

b. $|x^2 - 4| < 5$

This inequality is equivalent to $-5 < x^2 - 4 < 5$, so $-1 < x^2 < 9$. Hence

$$-3 < x < 3$$

c. $|x| + |x - 1| < 3$

Split the solution into three cases.

- Case 1: $x \leq 0$

For $x \leq 0$, the inequality becomes $-x - (x - 1) < 3$, so $-2x + 1 < 3$, hence $x > -1$. So, the interval $(-1, 0]$ is part of the solution.

- Case 2: $0 < x \leq 1$

For $0 < x \leq 1$, the inequality becomes $x - (x - 1) < 3$. This is equivalent to $1 < 3$, which is satisfied for all x . So, the interval $(0, 1]$ is also part of the solution.

- Case 3: $x > 1$

For $x > 1$, the inequality becomes $x + (x - 1) < 3$. This is equivalent to $2x - 1 < 3$, hence $x < 2$. So, the interval $(1, 2)$ is the final part of the solution.

From cases 1, 2, and 3 we conclude that the inequality $|x| + |x - 1| < 3$, is satisfied for all x in

$$(-1, 0] \cup (0, 1] \cup (1, 2) = (-1, 2)$$

8. For each of the following sequences, prove, using an ε, n_0 argument that the sequence converges to the given limit p ; that is, given $\varepsilon > 0$, determine n_0 such that $|p_n - p| < \varepsilon$ for all $n \geq n_0$.

b. $\{p_n\} = \left\{ \frac{2n-5}{6n-3} \right\}, p = \frac{1}{3}$

Let $\varepsilon > 0$. Observe that for all $n \in \mathbb{N}$

$$\left| \frac{2n-5}{6n-3} - \frac{1}{3} \right| = \frac{4}{3|2n-1|} = \frac{4}{3(n+(n-1))} \leq \frac{4}{3n}$$

Let n_0 denote a positive integer with $n_0 > \frac{4}{3\varepsilon}$. Then for all $n \geq n_0$

$$\left| \frac{2n-5}{6n-3} - \frac{1}{3} \right| \leq \frac{4}{3n} \leq \frac{4}{3n_0} < \frac{4}{3\left(\frac{4}{3\varepsilon}\right)} = \varepsilon$$

d. $\{p_n\} = \left\{ 1 - \frac{(-1)^n}{n} \right\}, p = 1$

Let $\varepsilon > 0$. Observe that for all $n \in \mathbb{N}$

$$\left| \left(1 - \frac{(-1)^n}{n} \right) - 1 \right| = \frac{1}{|n|} = \frac{1}{n}$$

Let n_0 denote a positive integer with $n_0 > \frac{1}{\varepsilon}$. Then for all $n \geq n_0$

$$\left| \left(1 - \frac{(-1)^n}{n} \right) - 1 \right| = \frac{1}{n} \leq \frac{1}{n_0} < \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon$$

9. Show that each of the following sequences diverge in \mathbb{R} .

c. $\{p_n\} = \left\{ \sin \frac{n\pi}{2} \right\}$

Observe that the distance between two consecutive elements p_n and p_{n+1} equals 1. Suppose that the sequence $\{p_n\}$ converges to p . Then there exists an n_0 such that for all $n \geq n_0$

$$|p_n - p| < \frac{1}{3}$$

Using the triangle inequality we obtain for all $n \geq n_0$

$$1 = |p_{n+1} - p_n| = |(p_{n+1} - p) + (p - p_n)| \leq |p_{n+1} - p| + |p - p_n| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

a contradiction.

d. $\{p_n\} = \left\{ n \sin \frac{n\pi}{2} \right\}$

Observe that the distance between two consecutive elements p_n and p_{n+1} is always greater than, or equal to, n . Suppose that the sequence $\{p_n\}$ converges to p . Then there exists an n_0 such that for all $n \geq n_0$

$$|p_n - p| < \frac{1}{3}$$

Using the triangle inequality we obtain for all $n \geq n_0$

$$1 \leq n \leq |p_{n+1} - p_n| = |(p_{n+1} - p) + (p - p_n)| \leq |p_{n+1} - p| + |p - p_n| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

a contradiction.

13. Let $\{a_n\}$ be a sequence in \mathbb{R} with $\lim_{n \rightarrow \infty} a_n = a$. Prove that $\lim_{n \rightarrow \infty} a_n^3 = a^3$.

Let $\varepsilon > 0$. Consider

$$|a_n^3 - a^3| = |a_n^2 + a_n a + a^2| |a_n - a| \leq \left(|a_n|^2 + |a_n| |a| + |a|^2 \right) |a_n - a|$$

Since the sequence $\{a_n\}$ converges, there exists a number $M > 0$, such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$

$$|a_n^3 - a^3| \leq \left(M^2 + M |a| + |a|^2 \right) |a_n - a|$$

Moreover, since $\lim_{n \rightarrow \infty} a_n = a$, there exists an $n_0 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{M^2 + M|a| + |a|^2}$ for all $n \geq n_0$. Therefore

$$|a_n^3 - a^3| \leq (M^2 + M|a| + |a|^2) |a_n - a| < (M^2 + M|a| + |a|^2) \frac{\varepsilon}{M^2 + M|a| + |a|^2} = \varepsilon$$

for all $n \geq n_0$. This completes the proof.

14. If $a_n \geq 0$ for all n and $\lim_{n \rightarrow \infty} a_n = a$, prove that $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$.

- Part 1: $a = 0$

Let $\varepsilon > 0$. Consider $|\sqrt{a_n} - \sqrt{0}| = |\sqrt{a_n}|$. Because $\lim_{n \rightarrow \infty} a_n = a = 0$, there exists an $n_0 \in \mathbb{N}$ such that $|a_n| < \varepsilon^2$ for all $n \geq n_0$. Hence, for all $n \geq n_0$,

$$|\sqrt{a_n} - \sqrt{0}| = |\sqrt{a_n}| < |\sqrt{\varepsilon^2}| = \varepsilon$$

This shows that $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{0} = \sqrt{a}$.

- Part 2: $a > 0$.

Let $\varepsilon > 0$. Consider $|\sqrt{a_n} - \sqrt{a}|$. In order to make use of the fact that $\lim_{n \rightarrow \infty} a_n = a$, we will need to rewrite the expression $|\sqrt{a_n} - \sqrt{a}|$ in a way that incorporates $|a_n - a|$. We do that as follows:

$$|\sqrt{a_n} - \sqrt{a}| = \left| (\sqrt{a_n} - \sqrt{a}) \cdot \frac{\sqrt{a_n} + \sqrt{a}}{\sqrt{a_n} + \sqrt{a}} \right| = \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| \leq \left| \frac{a_n - a}{\sqrt{a}} \right| = \frac{|a_n - a|}{\sqrt{a}}$$

Because $\lim_{n \rightarrow \infty} a_n = a$, there exists an $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon\sqrt{a}$ for all $n \geq n_0$. Hence, for all $n \geq n_0$

$$|\sqrt{a_n} - \sqrt{a}| \leq \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon$$

This shows once again that $\lim_{n \rightarrow \infty} \sqrt{a_n} = \sqrt{a}$.

15. Prove that if $\{a_n\}$ converges to a , then $\{|a_n|\}$ converges to $|a|$.

Let $\varepsilon > 0$. Then there exist an $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq n_0$. Therefore

$$||a_n| - |a|| \leq |a_n - a| < \varepsilon$$

for all $n \geq n_0$. We conclude that $\{|a_n|\}$ converges to $|a|$.

The converse is not true. Just let $a = -1$ and $a_n = 1$ for all $n \in \mathbb{N}$. Then $\{|a_n|\}$ converges to $|a|$, but $\{a_n\}$ does not converge to a .

2.2 Limit Theorems

1. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences of real numbers with

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b$$

then

- a. Show that $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$

Let $\varepsilon > 0$. Consider

$$|a_n + b_n - (a + b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b|$$

Because $\lim_{n \rightarrow \infty} a_n = a$, there exists an $n_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq n_1$. Similarly, there exists an $n_2 \in \mathbb{N}$ such that $|b_n - b| < \frac{\varepsilon}{2}$ for all $n \geq n_2$. Let $n_3 = \max\{n_1, n_2\}$. Then, for all $n \geq n_3$

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof.

2. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- a. If $\{a_n\}$ and $\{a_n + b_n\}$ both converge, prove that the sequence $\{b_n\}$ converges.

Let $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} (a_n + b_n) = s$. Then, by Corollary 2.2.2, Part b, with $c = -1$, $\lim_{n \rightarrow \infty} (-a_n) = -a$. Hence, by Theorem 2.2.1, Part a

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} [-a_n + (a_n + b_n)] = -a + s$$

So, the sequence $\{b_n\}$ converges.

- b. Suppose $b_n \neq 0$ for all $n \in \mathbb{N}$. If $\{b_n\}$ and $\{a_n/b_n\}$ both converge, prove that the sequence $\{a_n\}$ also converges.

Let $\lim_{n \rightarrow \infty} b_n = b$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = q$. Then by Theorem 2.2.1, Part b

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \cdot b_n \right) = qb$$

So, the sequence $\{a_n\}$ converges.

3. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If $\{b_n\}$ is bounded and $\lim_{n \rightarrow \infty} a_n = 0$, then show that

$$\lim_{n \rightarrow \infty} a_n b_n = 0$$

Let $\varepsilon > 0$. Since $\{b_n\}$ is bounded, there exists an $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Therefore

$$|a_n b_n - 0| = |a_n b_n| = |a_n| |b_n| \leq |a_n| M$$

Moreover, since $\lim_{n \rightarrow \infty} a_n = 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $|a_n| < \frac{\varepsilon}{M}$. Hence, for all $n \geq n_0$

$$|a_n b_n - 0| \leq |a_n| M < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

This proves that $\lim_{n \rightarrow \infty} a_n b_n = 0$.

5. a. If $p > 0$, prove that $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$.

Let $\varepsilon > 0$. Observe $\left| \frac{1}{n^p} \right| = \frac{1}{n^p}$. Now choose $n_0 \in \mathbb{N}$ with $n_0 > (1/\varepsilon)^{1/p}$. Then for all $n \geq n_0$

$$\left| \frac{1}{n^p} \right| = \frac{1}{n^p} \leq \frac{1}{n_0^p} < \frac{1}{\left[(1/\varepsilon)^{1/p} \right]^p} = \frac{1}{(1/\varepsilon)} = \varepsilon$$

This completes the proof.

- b. If $p > 0$, prove that $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$.

Make a distinction between $p > 1$, $p = 1$, and $0 < p < 1$.

- i. If $p > 1$, then let $x_n = \sqrt[p]{p} - 1$. Observe that $x_n > 0, \forall n \in \mathbb{N}$. Using the binomial theorem we obtain for $n \geq 1$

$$p = (1 + x_n)^n = \sum_{k=0}^n \binom{n}{k} x_n^k \geq \binom{n}{1} x_n = n x_n$$

so

$$0 \leq x_n \leq \frac{p}{n}$$

According to the squeeze theorem this implies that $\lim_{n \rightarrow \infty} x_n = 0$. Therefore $\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1$.

- ii. If $p = 1$ the result is trivial, because in this case

$$\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} \sqrt[1]{1} = \lim_{n \rightarrow \infty} 1 = 1$$

- iii. If $0 < p < 1$, write $q = \frac{1}{p}$. Then $q > 1$, and by Part i. we may conclude that $\lim_{n \rightarrow \infty} \sqrt[p]{q} = 1$. So, by Theorem 2.2.1 Part c,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[p]{q}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[p]{q}}$$

Hence

$$\lim_{n \rightarrow \infty} \sqrt[p]{p} = \lim_{n \rightarrow \infty} \sqrt[p]{\frac{1}{q}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[p]{q}} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[p]{q}} = \frac{1}{1} = 1$$

This completes the desired proof.

6. Find the limit of each of the following sequences.

a. $\left\{ \frac{3n^2 + 2n + 1}{5n^2 - 2n + 3} \right\}_{n=1}^{\infty}$

Factor out an n^2 from both the numerator and the denominator

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 2n + 1}{5n^2 - 2n + 3} = \lim_{n \rightarrow \infty} \frac{n^2 \left(3 + \frac{2}{n} + \frac{1}{n^2} \right)}{n^2 \left(5 - \frac{2}{n} + \frac{3}{n^2} \right)} = \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n} + \frac{1}{n^2}}{5 - \frac{2}{n} + \frac{3}{n^2}} = \frac{3}{5}$$

d. $\left\{ \frac{2\sqrt{n}}{n+1} \right\}_{n=1}^{\infty}$

Factor out an n from both the numerator and the denominator

$$\lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{n+1} = \lim_{n \rightarrow \infty} \frac{n \left(\frac{2}{\sqrt{n}} \right)}{n \left(1 + \frac{1}{n} \right)} = \lim_{n \rightarrow \infty} \frac{\frac{2}{\sqrt{n}}}{1 + \frac{1}{n}} = \frac{0}{1} = 0$$

f. $\left\{ \sqrt{n^2 + n} - n \right\}_{n=1}^{\infty}$

To analyze the cancellation effect in the general term of this sequence, we employ the well-known identity

$$(a - b)(a + b) = a^2 - b^2$$

Observe

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\sqrt{n^2 + n} - n \right) &= \lim_{n \rightarrow \infty} \left[\left(\sqrt{n^2 + n} - n \right) \cdot \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} \right] = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n \left(\sqrt{1 + \frac{1}{n}} + 1 \right)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2} \end{aligned}$$

h. $\left\{ (2^n + 3^n)^{1/n} \right\}_{n=1}^{\infty}$

First, factor out the leading term

$$\lim_{n \rightarrow \infty} (2^n + 3^n)^{1/n} = 3 \lim_{n \rightarrow \infty} \left(\left(\frac{2}{3} \right)^n + 1 \right)^{1/n}$$

Observe that the sequence $\left\{ \left(\frac{2}{3} \right)^n + 1 \right\}$ is bounded. Therefore, in light of Theorem 2.2.1, Part b, one might expect that

$$\lim_{n \rightarrow \infty} \left(\left(\frac{2}{3} \right)^n + 1 \right)^{1/n} = 1$$

This proposition is readily proved using the squeeze theorem. Clearly, for all $n \in \mathbb{N}$

$$1 \leq \left(\left(\frac{2}{3} \right)^n + 1 \right)^{1/n} \leq \left(\frac{5}{3} \right)^{1/n}$$

Because $\lim_{n \rightarrow \infty} \left(\frac{5}{3} \right)^{1/n} = 1$, it follows that $\lim_{n \rightarrow \infty} \left(\left(\frac{2}{3} \right)^n + 1 \right)^{1/n} = 1$ and

$$\lim_{n \rightarrow \infty} (2^n + 3^n)^{1/n} = 3 \lim_{n \rightarrow \infty} \left(\left(\frac{2}{3} \right)^n + 1 \right)^{1/n} = 3$$

7. d. This exercise was solved in class.
f. This exercise was solved in class.

2.3 Monotone Sequences

1. This exercise was solved in class.
3. Show that each of the following sequences are monotone. Find an upper or lower bound if it exists; find the limit if you can.

a. $\left\{ \frac{\sqrt{n^2+1}}{n} \right\}$

Let $a_n = \frac{\sqrt{n^2+1}}{n}$, then

$$a_n = \frac{\sqrt{n^2+1}}{n} = \sqrt{1 + \frac{1}{n^2}}$$

This shows that the sequence is monotone decreasing. Moreover, $1 \leq \sqrt{1 + \frac{1}{n^2}} \leq \sqrt{2}$, so the sequence is monotone and bounded. Hence, it converges, and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n^2}} = 1$$

5. Let $a_1 > 1$, and for $n \in \mathbb{N}$, define $a_{n+1} = 2 - 1/a_n$. Show that the sequence $\{a_n\}$ is monotone and bounded. Find $\lim_{n \rightarrow \infty} a_n$.

Follow the hints in the back of the book. First, use mathematical induction to show that the sequence is bounded below by 1, then prove that the sequence is monotone decreasing.

- It is given that $a_1 > 1$.
- Let $k \in \mathbb{N}$ and suppose $a_k > 1$. Then $1/a_k < 1$ and $-1/a_k > -1$, so

$$a_{k+1} = 2 - 1/a_k > 2 - 1 = 1$$

This establishes the fact that the sequence is bounded below by 1.

To show that the sequence is monotone decreasing, we use the inequality $0 \leq (1 - a_n)^2 = 1 - 2a_n + a_n^2$, which implies $a_n^2 \geq 2a_n - 1$, so

$$a_n \geq 2 - 1/a_n = a_{n+1}$$

This proves that the sequence is monotone decreasing. Combination of this result with the above derived lower bound, shows that

$$1 \leq a_n \leq a_1$$

Hence, the sequence is monotone and bounded, so it converges. Let $\lim_{n \rightarrow \infty} a_n = a$, then

$$a = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} (2 - 1/a_n) = 2 - 1/a$$

This yields $a^2 - 2a + 1 = 0$, so $a = 1$.

7. For each of the following, prove that the sequence $\{a_n\}$ converges and find the limit.

a. $a_{n+1} = \frac{1}{6}(2a_n + 5)$, $a_1 = 2$

To get a feeling for what the sequence is doing, compute a decimal approximation of the first couple of terms.

$$\{2.0, 1.50000, 1.33333, 1.27778, 1.25926, 1.25309, 1.25103, 1.25034, 1.25011, 1.25004, \dots\}$$

We suspect that the sequence is monotone decreasing and bounded below by $5/4$. Both propositions will be proved by mathematical induction. First, we establish the lower bound of $5/4$.

- $a_1 = 2 \geq 5/4$.

- Suppose $k \in \mathbb{N}$ and $a_k \geq 5/4$. Then

$$a_{k+1} = \frac{1}{6}(2a_k + 5) \geq \frac{1}{6}\left(2 \cdot \frac{5}{4} + 5\right) = \frac{5}{4}$$

Hence, $a_n \geq 5/4$ for all $n \in \mathbb{N}$.

Next, we establish the fact that the sequence $\{a_n\}$ is monotone decreasing.

- $a_1 = 2 \geq 3/2 = a_2$.
- Suppose $k \in \mathbb{N}$ and $a_k \geq a_{k+1}$. Then

$$a_{k+1} = \frac{1}{6}(2a_k + 5) \geq \frac{1}{6}(2a_{k+1} + 5) = a_{k+2}$$

Hence, $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Since the sequence $\{a_n\}$ is monotone and bounded, it must be convergent. Let

$$\lim_{n \rightarrow \infty} a_n = a$$

then

$$a = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{6}(2a_n + 5) = \frac{1}{6}(2a + 5)$$

Therefore, $a = \frac{1}{6}(2a + 5)$, which implies that $a = 5/4$, a result that is consistent with the numerical information found earlier.

12. Use Example 2.3.5 to find the limit of each of the following sequences.

a. $\left\{ \left(1 + \frac{1}{n}\right)^{2n} \right\}$

Observe

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{2n} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n \right] = e \cdot e = e^2$$

d. $\left\{ \left(1 - \frac{1}{n}\right)^n \right\}$

The solution to this problem is not as simple as that of Part a. To make use of the fact that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$, we rewrite $\left(1 - \frac{1}{n}\right)^n$ as

$$\left(1 - \frac{1}{n}\right)^n \cdot \frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 + \frac{1}{n}\right)^n} = \frac{\left(1 - \frac{1}{n^2}\right)^n}{\left(1 + \frac{1}{n}\right)^n}$$

Next, we prove that $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right)^n = 1$. We apply an estimation process similar to that used in Example 2.3.5 of the textbook.

Let $\varepsilon > 0$. Consider

$$\begin{aligned} \left| \left(1 - \frac{1}{n^2}\right)^n - 1 \right| &= \left| \sum_{k=0}^n \binom{n}{k} \left(-\frac{1}{n^2}\right)^k - 1 \right| = \left| \sum_{k=1}^n \binom{n}{k} \frac{(-1)^k}{n^{2k}} \right| \leq \sum_{k=1}^n \binom{n}{k} \frac{1}{n^{2k}} \\ &= \sum_{k=1}^n \frac{(n-k+1)(n-k+2)\cdots(n-1)n}{k!} \cdot \frac{1}{n^{2k}} \leq \sum_{k=1}^n \frac{1}{k!n^k} \\ &\leq \frac{1}{n} \sum_{k=1}^n \frac{1}{k!} \leq \frac{1}{n} \sum_{k=1}^n \frac{1}{2^{k-1}} = \frac{1}{n} \cdot \frac{1 - (1/2)^n}{1 - 1/2} \\ &= \frac{1}{n} \left(2 - (1/2)^{n-1}\right) \leq \frac{2}{n} \end{aligned}$$

Let $n_0 > 2/\varepsilon$. Then for all $n \in \mathbb{N}$ with $n \geq n_0$

$$\left| \left(1 - \frac{1}{n^2}\right)^n - 1 \right| \leq \frac{2}{n} \leq \frac{2}{n_0} < \frac{2}{2/\varepsilon} = \varepsilon$$

This proves that $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n^2}\right)^n = 1$. Therefore

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} \frac{\left(1 - \frac{1}{n^2}\right)^n}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} = e^{-1}$$

13. This exercise was solved in class.
14. This exercise was solved in class.
17. Show that each of the following sequences diverges to ∞ .

b. $\{a_n\} = \left\{n + \frac{(-1)^n}{n}\right\}$

Let $M > 0$. Observe that for all $n \in \mathbb{N}$

$$a_n = n + \frac{(-1)^n}{n} \geq n - 1$$

Choose $n_0 > M + 1$, then for all $n \geq n_0$

$$a_n = n + \frac{(-1)^n}{n} \geq n - 1 \geq n_0 - 1 > (M + 1) - 1 = M$$

This completes the proof.

c. $\{a_n\} = \{n + (-1)^n \sqrt{n}\}$

Let $M > 0$. Observe

$$a_n = n + (-1)^n \sqrt{n} = \frac{n}{2} + \left(\frac{n}{2} + (-1)^n \sqrt{n}\right) = \frac{n}{2} + \frac{1}{2} \sqrt{n} (\sqrt{n} + (-1)^n 2)$$

This shows that $a_n \geq \frac{n}{2}$ for all $n \geq 4$. Choose $n_0 > \max\{4, 2M\}$, then for all $n \geq n_0$

$$a_n \geq \frac{n}{2} \geq \frac{n_0}{2} > \frac{2M}{2} = M$$

This completes the proof.

18. Which of the sequences in the previous exercise are monotone? Explain your answer.

b. $\{a_n\} = \left\{n + \frac{(-1)^n}{n}\right\}$

Observe that $a_2 = \frac{5}{2} \geq 0 = a_1$, and for all $n \geq 2$

$$\begin{aligned} a_{n+1} - a_n &= n + 1 + \frac{(-1)^{n+1}}{n+1} - \left(n + \frac{(-1)^n}{n}\right) = 1 + \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n} \\ &\geq 1 - \frac{1}{n+1} - \frac{1}{n} \geq 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6} \geq 0 \end{aligned}$$

We conclude that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, so the sequence is monotone increasing.

c. $\{a_n\} = \{n + (-1)^n \sqrt{n}\}$

Observe $a_1 = 0$, $a_2 = 2 + \sqrt{2} \approx 3.4142$, $a_3 = 3 - \sqrt{3} \approx 1.2679$. Hence, the sequence is definitely not monotone.

More important for practical purposes, is to observe that the sequence will never become monotone, even for large values of n . To understand this phenomenon, we investigate the differences $a_{2n} - a_{2n-1}$ and $a_{2n+1} - a_{2n}$. Observe that for all $n \in \mathbb{N}$

$$a_{2n} - a_{2n-1} = 2n + \sqrt{2n} - (2n - 1 - \sqrt{2n - 1}) = \sqrt{2n - 1} + \sqrt{2n} + 1 > 0$$

and

$$a_{2n+1} - a_{2n} = 2n + 1 - \sqrt{2n + 1} - (2n + \sqrt{2n}) = 1 - \sqrt{2n} - \sqrt{2n + 1} < 0$$

Hence, $a_{2n} > a_{2n-1}$ while $a_{2n+1} < a_{2n}$. In words: every even indexed term is greater than its predecessor, while every odd indexed term is smaller than its predecessor.

2.4 Subsequences and the Bolzano-Weierstrass Theorem

1. b. Let $p_n = n(1 + (-1)^n)$. Prove that 0 and ∞ are the only subsequential limits of the sequence $\{p_n\}$.

Because

$$\lim_{k \rightarrow \infty} p_{2k} = \lim_{k \rightarrow \infty} \left[2k \left(1 + (-1)^{2k} \right) \right] = \lim_{k \rightarrow \infty} 4k = \infty$$

and

$$\lim_{k \rightarrow \infty} p_{2k-1} = \lim_{k \rightarrow \infty} \left[(2k-1) \left(1 + (-1)^{2k-1} \right) \right] = \lim_{k \rightarrow \infty} 0 = 0$$

0 and ∞ are subsequential limits of the sequence $\{p_n\}$. To show they are the only subsequential limits of $\{p_n\}$, we use the same technique as applied in class. Let $\{p_{n_k}\}$ denote any subsequence of $\{p_n\}$, then

- if the sequence $\{n_k\}$ has only finitely many odd elements, we conclude that

$$\lim_{k \rightarrow \infty} p_{n_k} = \infty$$

- if the sequence $\{n_k\}$ has only finitely many even elements, we conclude that

$$\lim_{k \rightarrow \infty} p_{n_k} = 0$$

- if the sequence $\{n_k\}$ has infinitely many odd elements and infinitely many even elements, then the sequence $\{p_{n_k}\}$ has a subsequence that converges to 0, and it has a subsequence that diverges to ∞ , so the sequence $\{p_{n_k}\}$ itself cannot converge.

Hence, 0 and ∞ are the only subsequential limits of $\{n(1 + (-1)^n)\}$.

2. a. Construct a sequence $\{s_n\}$ for which the subsequential limits are $\{-\infty, -2, 1\}$.

For all $k \in \mathbb{N}$ let

$$\begin{aligned} s_{3k-2} &= -k \\ s_{3k-1} &= -2 \\ s_{3k} &= 1 \end{aligned}$$

- b. Construct a sequence $\{s_n\}$ for which the set of subsequential limits of the sequence is countable.

Take for instance the sequence $\{s_n\}$ given by

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 7, 1, \dots$$

Observe that ∞ , and every $n \in \mathbb{N}$ is a subsequential limit of this sequence. The set $\{\infty\} \cup \mathbb{N}$ is countable.

3. Find all the subsequential limits of each of the following sequences.

a. $\left\{\sin \frac{n\pi}{2}\right\}$

Observe that

$$\left\{\sin \frac{n\pi}{2}\right\} = 1, 0, -1, 0, 1, 0, -1, 0, \dots$$

So, $-1, 0$, and 1 are subsequential limits of the sequence $\left\{\sin \frac{n\pi}{2}\right\}$. To be precise, if $p_n = \sin \frac{n\pi}{2}$, then

- $\lim_{k \rightarrow \infty} p_{4k-1} = -1$
- $\lim_{k \rightarrow \infty} p_{4k-2} = 0$
- $\lim_{k \rightarrow \infty} p_{4k-3} = 1$

Moreover, they are the only subsequential limits of that sequence, because the range of $\{p_n\}$ contains only these three numbers.

c. $\{p_n\} = \left\{(-1)^n + 2 \sin \frac{n\pi}{2}\right\}$

Observe that,

$$\{p_n\} = \left\{(-1)^n + 2 \sin \frac{n\pi}{2}\right\} = 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, 1, 1, -3, 1, \dots$$

An argument similar to that applied in Part a, shows that -3 and 1 , are the only subsequential limits of $\left\{(-1)^n + 2 \sin \frac{n\pi}{2}\right\}$.

7. Determine the limit points and the isolated points of each of the following sets.

b. $E = \left\{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\right\}$

The set E has two limit points, -1 and 1 . Every point of E is an isolated point of E .

c. $E = (0, 1) \cup \{2\}$

The set E has one isolated point, 2 . The set of limit points of E is the closed interval $[0, 1]$.

10. a. Construct a subset of \mathbb{R} with exactly two limit points.

Use the set E in Exercise 7b.

b. Find an infinite subset of \mathbb{R} with no limit points.

\mathbb{N} is an infinite subset of \mathbb{R} with no limit points.

c. Construct a countable subset of \mathbb{R} with countably many limit points.

For all $m \in \mathbb{N}$, let

$$E_m = \left\{2m + \frac{1}{n} : n \in \mathbb{N}\right\}$$

and let

$$S = \cup_{m \in \mathbb{N}} E_m$$

Then S is a countable subset of \mathbb{R} with countably many limit points.

d. Find a countable subset of \mathbb{R} with uncountably many limit points.

\mathbb{Q} is a countable subset of \mathbb{R} with uncountably many limit points.

11. Prove Theorem 2.4.12. Let $\{p_n\}$ be a sequence in \mathbb{R} . If p is a limit point of $\{p_n : n \in \mathbb{N}\}$, then there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \rightarrow p$ as $k \rightarrow \infty$.

Apply the same argument as used in the proof of Corollary 2.4.11. Choose n_1 such that $|p_{n_1} - p| < 1$. Having chosen n_1, n_2, \dots, n_{k-1} , choose an integer $n_k > n_{k-1}$ so that

$$|p_{n_k} - p| < \frac{1}{k}$$

Such an integer n_k exists since every neighborhood of p contains infinitely many points of $\{p_n : n \in \mathbb{N}\}$. By construction, the sequence $\{p_{n_k}\}_{k=1}^{\infty}$ converges to p .

12. Prove that every sequence in \mathbb{R} has a monotone subsequence.

Let $\{p_n\}$ be a sequence in \mathbb{R} .

- If $\{p_n\}$ is unbounded, then the range of $\{p_n\}$ is unbounded. Suppose the range of $\{p_n\}$ is not bounded above. Choose n_1 such that $p_{n_1} > 1$. Having chosen n_1, n_2, \dots, n_{k-1} , choose an integer $n_k > n_{k-1}$ so that

$$p_{n_k} > p_{n_{k-1}}$$

By construction, the sequence $\{p_{n_k}\}$ is monotone increasing. In case the range of $\{p_n\}$ is not bounded below, a similar argument allows for the construction of a monotone decreasing sequence $\{p_{n_k}\}$.

- If $\{p_n\}$ is bounded, then the range of $\{p_n\}$ is bounded. We now distinguish two cases.
 - Case 1: The range of $\{p_n\}$ is finite. Then there exists a point $p \in \{p_n : n \in \mathbb{N}\}$ and a sequence $\{n_k\}$, with $n_1 < n_2 < \dots$ such that

$$p_{n_1} = p_{n_2} = \dots = p$$

Clearly, the sequence $\{p\}$ is a monotone subsequence of $\{p_n\}$.

- Case 2: The range of $\{p_n\}$ is infinite. Then since $\{p_n : n \in \mathbb{N}\}$ is bounded and infinite, it must have a limit point p . By Theorem 2.4.12, there exists a subsequence $\{p_{n_k}\}$ that converges to p .

- * If $p_{n_k} = p$ for infinitely many $k \in \mathbb{N}$, then the sequence $\{p\}$ is a monotone subsequence of $\{p_n\}$.
- * If $p_{n_k} = p$ for only finitely many $k \in \mathbb{N}$, then delete from $\{p_{n_k}\}$ those elements that equal p , and, for convenience of notation, denote the resulting subsequence of $\{p_n\}$ by $\{q_k\}$. Hence,

$$\lim_{k \rightarrow \infty} q_k = p \text{ and } q_k \neq p \text{ for all } k \in \mathbb{N}$$

Since $N_1(p) \cap \{q_k : k \in \mathbb{N}\}$ is infinite, either the interval $(p, p+1)$, or the interval $(p-1, p)$ must contain infinitely many points of $\{q_k : k \in \mathbb{N}\}$. Suppose $(p, p+1) \cap \{q_k : k \in \mathbb{N}\}$ is infinite. Choose k_1 such that $p < q_{k_1} < p+1$. Having chosen k_1, k_2, \dots, k_{m-1} , choose an integer $k_m > k_{m-1}$ so that

$$p < q_{k_m} < q_{k_{m-1}}$$

By construction $\{q_{k_m}\}$ is a monotone decreasing subsequence of $\{p_n\}$. In case $(p-1, p) \cap \{q_k : k \in \mathbb{N}\}$ is infinite, a similar argument allows for the construction of a monotone increasing subsequence of $\{p_n\}$.

2.5 Limit Superior and Inferior of a Sequence

2.6 Cauchy Sequences

3. For $n \in \mathbb{N}$, let

$$s_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Prove that $\{s_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$. We use a technique similar to that used in Example 2.3.5, and estimate $|s_{n+k} - s_n|$ by a partial sum of a geometric series.

$$\begin{aligned} |s_{n+k} - s_n| &= \sum_{m=1}^k \frac{1}{(n+m)!} \leq \sum_{m=1}^k \frac{1}{2^{n+m-1}} = \frac{1}{2^n} \sum_{m=1}^k \frac{1}{2^{m-1}} \\ &= \frac{1}{2^n} \cdot \frac{1 - \left(\frac{1}{2}\right)^k}{1 - \frac{1}{2}} \leq \frac{1}{2^{n-1}} \leq \frac{1}{n} \end{aligned}$$

Choose $n_0 > \frac{1}{\varepsilon}$, then for all $n \geq n_0$ and for all $k \in \mathbb{N}$

$$|s_{n+k} - s_n| \leq \frac{1}{n} \leq \frac{1}{n_0} = \frac{1}{1/\varepsilon} = \varepsilon$$

This completes the proof.

4. Consider the sequence $\{s_n\}$ defined for $n = 1, 2, 3, \dots$ by

$$s_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

- a. Show that $\{s_n\}$ is not a Cauchy sequence.

We proved in class that the sequence $\{s_n\}$ diverges to infinity. (Exercise 13, Section 2.3) Hence, $\{s_n\}$ is not a Cauchy sequence.

- b. Even though $\{s_n\}$ is not a Cauchy sequence, show that $\lim_{n \rightarrow \infty} |s_{n+k} - s_n| = 0$ for all $k \in \mathbb{N}$.

Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Choose $n_0 > \frac{k}{\varepsilon}$, then for all $n \geq n_0$

$$|s_{n+k} - s_n| = \left| \sum_{m=1}^k \frac{1}{n+m} \right| \leq \frac{k}{n} \leq \frac{k}{n_0} < \frac{k}{k/\varepsilon} = \varepsilon$$

This shows that $\lim_{n \rightarrow \infty} |s_{n+k} - s_n| = 0$ for all $k \in \mathbb{N}$.

9. Let $c_1 \in (0, 1)$ be arbitrary, and for $n \in \mathbb{N}$ set $c_{n+1} = \frac{1}{5}(c_n^2 + 2)$.

- a. Show that the sequence $\{c_n\}$ is contractive.

Observe that if $c_n \in (0, 1)$, then

$$0 < \frac{2}{5} < c_{n+1} < \frac{3}{5} < 1$$

Therefore, since $c_1 \in (0, 1)$, we may conclude that $c_n \in (0, 1)$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} |c_{n+1} - c_n| &= \left| \frac{1}{5}(c_n^2 + 2) - \frac{1}{5}(c_{n-1}^2 + 2) \right| = \frac{1}{5} |c_n^2 - c_{n-1}^2| \\ &= \frac{1}{5} |c_n + c_{n-1}| |c_n - c_{n-1}| \leq \frac{2}{5} |c_n - c_{n-1}| \end{aligned}$$

so the sequence $\{c_n\}$ is contractive with $b = \frac{2}{5}$.

- b. Let $c = \lim_{n \rightarrow \infty} c_n$. Show that c is a solution to $x^2 - 5x + 2 = 0$.

Take the limit as $n \rightarrow \infty$ of both sides of the recurrence relation $c_{n+1} = \frac{1}{5}(c_n^2 + 2)$.

$$c = \lim_{n \rightarrow \infty} c_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{5}(c_n^2 + 2) = \frac{1}{5}(c^2 + 2)$$

so $c^2 - 5c + 2 = 0$.

- c. Let $c_1 = \frac{1}{2}$. Using the result of Theorem 2.6.7(b), determine the value of n such that $|c - c_n| < 10^{-3}$.

The iteration results with the error bounds are listed in the table below.

n	c_n	$\frac{b}{1-b} c_n - c_{n-1} $
1	0.5000000000	
2	0.4500000000	0.033333333334
3	0.4405000000	0.006333333334
4	0.4388080500	0.001127966667
5	0.4385105009	0.000198366034

Clearly, $n = 5$ will do the job. The value $c_5 = 0.4385105009$ is guaranteed to be less than 10^{-3} away from a solution of the equation $x^2 - 5x + 2 = 0$.