Chapter 3

Structure of Point Sets

3.1 Open and Closed Sets

6. a. This problem was solved in class.

   b. Give an example of a countable collection $\{F_n\}_{n=1}^{\infty}$ of closed subsets of $\mathbb{R}$ such that $\cup_{n=1}^{\infty} F_n$ is not closed.

   Take for instance $F_n = \left[\frac{1}{n+1}, 1 - \frac{1}{n+1}\right]$, then $\cup_{n=1}^{\infty} F_n = (0,1)$.

   Alternatively, we could have applied a De Morgan law to one of the examples presented during the class meeting of November 10.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0,1]$$

   Let $E_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ and $B = [0,1]$, then $\cup_{n=1}^{\infty} E_n^c = B^c$. Hence $\{E_n^c\}_{n=1}^{\infty}$ is an example of a countable collection of closed subsets of $\mathbb{R}$ such that $\cup_{n=1}^{\infty} E_n^c$ is open.

8. Let E be a subset of $\mathbb{R}$.

   a. Prove that $\operatorname{Int}(E)$ is open.

   Let $p \in \operatorname{Int}(E)$. Then there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \subset E$. Actually, $N_\varepsilon(p) \subset \operatorname{Int}(E)$. To show this we let $q \in N_\varepsilon(p)$. Since $N_\varepsilon(p)$ is open, there exists a $\delta > 0$ such that $N_\delta(q) \subset N_\varepsilon(p) \subset E$. So $q \in \operatorname{Int}(E)$ and therefore $N_\varepsilon(p) \subset \operatorname{Int}(E)$. This completes the proof.

   b. Prove that E is open if and only if $E = \operatorname{Int}(E)$.

   - If $E = \operatorname{Int}(E)$ then, by Part (a), E is open.

- Conversely, if E is open, then every point of E is an interior point of E. This means $E \subset \text{Int}(E)$. Moreover, $\text{Int}(E) \subset E$ for any set E.
  We conclude that $E = \text{Int}(E)$.

c. If $G \subset E$ and G is open, prove that $G \subset \text{Int}(E)$.

Let $p \in G$. Since G is open, there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \subset G \subset E$. Therefore $p \in \text{Int}(E)$, hence $G \subset \text{Int}(E)$.

This result tells us that $\text{Int}(E)$ is the largest open set contained in E.

10. Prove that the set of limit points of a set is closed.

Let $E \subset \mathbb{R}$ and let E' denote the set of limit points of E. Suppose $p \in (E')^c$. Then p is not a limit point of E, so there exists an $\varepsilon > 0$ such that $N_\varepsilon(p) \cap E$ contains at most p. We will use a proof by contradiction to show that $N_\varepsilon(p) \subset (E')^c$. Suppose that $q \in N_\varepsilon(p)$ is a limit point of E. Since $N_\varepsilon(p)$ is open, there exists a $\delta > 0$ such that $N_\delta(q) \subset N_\varepsilon(p)$. Because $q \in E'$, $N_\delta(q)$ contains infinitely many elements of E, hence $N_\varepsilon(p)$ contains infinitely many elements of E, a contradiction. We conclude that $N_\varepsilon(p) \subset (E')^c$, which means $(E')^c$ is open, so E' is closed.

11. Let $E \subset \mathbb{R}$. A point $p \in \mathbb{R}$ is a **boundary point** of E if for every $\varepsilon > 0$, $N_\varepsilon(p)$ contains both points of E and points of E^c. Find the boundary points of each of the following sets.

a. (a, b)

The set of boundary points of (a, b) equals $\{a, b\}$.

b. $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

The set of boundary points of E equals $E \cup \{0\}$.

c. $\mathbb{N}$

The set of boundary points of $\mathbb{N}$ equals $\mathbb{N}$.

d. $\mathbb{Q}$

The set of boundary points of $\mathbb{Q}$ equals $\mathbb{Q}$.