Chapter 4

Limits and Continuity

4.1 Limit of a Function

1. Use the definition to establish each of the following limits.

   a. $\lim_{x\to 2}(2x-7)=-3$
   Suppose $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x-7$. Clearly, 2 is a limit point of $\mathbb{R}$. Let $\varepsilon>0$. Observe that for all $x\in\mathbb{R}$

$$|f(x)-3|=|(2x-7)-(-3)|=|2x-4|=2|x-2|$$

   Choose $\delta=\frac{\varepsilon}{2}$. Then for all $x\in\mathbb{R}$ with $0<|x-2|<\delta$

$$|(2x-7)-(-3)|=2|x-2|<2\delta=2\frac{\varepsilon}{2}=\varepsilon$$

   This completes the proof.

   f. $\lim_{x\to 2}\frac{x^3-2x-4}{x^2-4}=\frac{5}{2}$
   Suppose $E=(-\infty,-2)\cup(-2,2)\cup(2,\infty)$ and $f:E\to\mathbb{R}$, $f(x)=\frac{x^3-2x-4}{x^2-4}$. Clearly, 2 is a limit point of E. Let $\varepsilon>0$. Observe that for all $x\in E$

$$\left|f(x)-\frac{5}{2}\right|=\left|\frac{x^3-2x-4}{x^2-4}-\frac{5}{2}\right|=\frac{1}{2}\left|\frac{2x^2-x-6}{x+2}\right|=\frac{1}{2}\left|\frac{2x+3}{x+2}\right||x-2|$$

   To estimate $\left|\frac{2x+3}{x+2}\right|$, we place an initial restriction on δ. Let $\delta\le 1$. Then for all $x\in E$ with $0<|x-2|<\delta\le 1$

$$1\le x\le 3$$

so

$$5 \leq 2x + 3 \leq 9 \text{ and } 3 \leq x + 2 \leq 5$$

thus $\left|\frac{2x+3}{x+2}\right| \leq \frac{9}{3} = 3.$

Choose $\delta = \min\left\{1, \frac{2\varepsilon}{3}\right\}$. Then for all $x \in E$ with $0 < |x - 2| < \delta$

$$\left|\frac{x^3 - 2x - 4}{x^2 - 4} - \frac{5}{2}\right| = \frac{1}{2}\left|\frac{2x+3}{x+2}\right| |x - 2| \leq \frac{3}{2}|x - 2| < \frac{3}{2}\delta \leq \frac{3}{2} \cdot \frac{2\varepsilon}{3} = \varepsilon$$

This completes the proof.

2. Use the definition to establish each of the following limits.

   c. $\lim_{x \to p} c = c$
   Suppose $p, c \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c$. Clearly, p is a limit point of $\mathbb{R}$. Let $\varepsilon > 0$. Observe that for all $x \in \mathbb{R}$

   $$|f(x) - c| = |c - c| = 0$$

   We conclude that any $\delta > 0$ will do the job. Take for instance $\delta = 1$. Then for all $x \in \mathbb{R}$ with $0 < |x - p| < \delta$

   $$|f(x) - c| = 0 < \varepsilon$$

   This completes the proof.

   e. $\lim_{x \to p} \sqrt{x} = \sqrt{p}$, $p > 0$.
   Suppose $E = [0, \infty)$ and $f : E \to \mathbb{R}$, $f(x) = \sqrt{x}$. Clearly, p is a limit point of E. Let $\varepsilon > 0$. Observe that for all $x \in E$

   $$\begin{aligned} |f(x) - \sqrt{p}| &= \left|\sqrt{x} - \sqrt{p}\right| = \left|\frac{x - p}{\sqrt{x} + \sqrt{p}}\right| \\ &= \frac{1}{\sqrt{x} + \sqrt{p}}|x - p| \leq \frac{1}{\sqrt{p}}|x - p| \end{aligned}$$

   Choose $\delta = \varepsilon\sqrt{p}$. Then for all $x \in E$ with $0 < |x - p| < \delta$

   $$|f(x) - \sqrt{p}| \leq \frac{1}{\sqrt{p}}|x - p| < \frac{1}{\sqrt{p}}\delta = \frac{1}{\sqrt{p}} \cdot \varepsilon\sqrt{p} = \varepsilon$$

   This completes the proof.

3. For each of the following, determine whether the indicated limit exists in $\mathbb{R}$. Justify your answer.

b. $\lim_{x\to 1} \frac{x^2-1}{x+1}$
We conjecture that this limit equals zero. Let $E = (-\infty, -1) \cup (-1, \infty)$ and $f : E \to \mathbb{R}, f(x) = \frac{x^2-1}{x+1}$. Suppose $\varepsilon > 0$. Observe that for all $x \in E$

$$\left|\frac{x^2-1}{x+1}\right| = |x-1|$$

So, if we let $\delta = \varepsilon$, then for all $x \in E$ with $0 < |x-1| < \delta = \varepsilon$

$$\left|\frac{x^2-1}{x+1}\right| = |x-1| < \delta = \varepsilon$$

This completes the proof.

e. $\lim_{x\to 1} \frac{x}{|x-1|}$
We conjecture that this limit does not exist in $\mathbb{R}$. Let $E = (-\infty, 1) \cup (1, \infty)$ and $f : E \to \mathbb{R}, f(x) = \frac{x}{|x-1|}$. Use the sequential criterion for limits and construct a sequence $\{p_n\}$ in E such that $p_n \neq 1$ for all $n \in \mathbb{N}$ and $p_n \to 1$ as $n \to \infty$, and such that $\{f(p_n)\}$ is unbounded. Take for instance

$$p_n = 1 + \frac{1}{n}$$

then

$$f(p_n) = \frac{1+\frac{1}{n}}{\left|1+\frac{1}{n}-1\right|} = n+1$$

Hence the sequence $\{f(p_n)\}$ is unbounded and cannot converge, so $\lim_{x\to 1} \frac{x}{|x-1|}$ does not exist in $\mathbb{R}$.

7. Suppose $f : E \to \mathbb{R}$. p is a limit point of E, and $\lim_{x\to p} f(x) = L$.

b. If, in addition $f(x) \geq 0$ for all $x \in E$, prove that $\lim_{x\to p} \sqrt{f(x)} = \sqrt{L}$.
Let $\varepsilon > 0$. Distinguish two cases

- $L = 0$
  Because $\lim_{x\to p} f(x) = 0$, there exists a $\delta > 0$ such that for all $x \in E$ with

$$0 < |x-p| < \delta$$

  $0 \leq f(x) = |f(x)| < \varepsilon^2$. Hence for those values of x

$$\left|\sqrt{f(x)}\right| = \sqrt{f(x)} < \sqrt{\varepsilon^2} = \varepsilon$$

  This shows $\lim_{x\to p} \sqrt{f(x)} = 0 = \sqrt{0} = \sqrt{L}$.

- $L > 0$
  Consider
  $$\left|\sqrt{f(x)} - \sqrt{L}\right| = \left|\frac{f(x) - L}{\sqrt{f(x)} + \sqrt{L}}\right| \leq \frac{1}{\sqrt{L}}\left|f(x) - L\right|$$
  Because $\lim_{x\to p} f(x) = 0$, there exists a $\delta > 0$ such that for all $x \in E$ with
  $$0 < |x - p| < \delta$$
  $|f(x) - L| < \varepsilon\sqrt{L}$. Hence for those values of x
  $$\left|\sqrt{f(x)} - \sqrt{L}\right| \leq \frac{1}{\sqrt{L}}\left|f(x) - L\right| < \frac{1}{\sqrt{L}} \cdot \varepsilon\sqrt{L} = \varepsilon$$
  This again shows $\lim_{x\to p} \sqrt{f(x)} = \sqrt{L}$.

8. Use the limit theorems, examples, and previous exercises to find each of the following limits. State which theorems, examples, or exercises are used in each case.

   h. $\lim_{x\to 0} \frac{|x-2|-|x+2|}{x}$
   A nice trick is to limit the domain of the quotient $\frac{|x-2|-|x+2|}{x}$ so that the expression can be easily simplified. To do that we bound x away from the points -2 and 2. Let $E = [-1, 1] \setminus \{0\}$ and $f : E \to \mathbb{R}$, $f(x) = \frac{|x-2|-|x+2|}{x}$. Then $0 \in E'$ and for all $x \in E$
   $$f(x) = \frac{|x-2| - |x+2|}{x} = \frac{(2-x) - (x+2)}{x} = \frac{-2x}{x} = -2$$
   Therefore
   $$\lim_{x\to 0} f(x) = \lim_{x\to 0} -2 = -2$$

13. Prove Theorem 4.1.9.
    Suppose $E \subset R$, $p \in E'$, $f, g, h : E \to R$ satisfying
    $$g(x) \leq f(x) \leq h(x) \text{ for all } x \in E$$
    If $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L$, then $\lim_{x\to p} f(x) = L$.
    Let $\varepsilon > 0$.

    - Since $\lim_{x\to p} g(x) = L$, there exists a $\delta_1 > 0$ such that for all $x \in E$ with
      $$0 < |x - p| < \delta_1$$
      $|g(x) - L| < \varepsilon$. So, for these values of x
      $$-\varepsilon < g(x) - L < \varepsilon$$

- Similarly, since $\lim_{x\to p} h(x) = L$, there exists a $\delta_2 > 0$ such that for all $x \in E$ with

$$0 < |x - p| < \delta_2$$

$|h(x) - L| < \varepsilon$. So, for these values of x

$$-\varepsilon < h(x) - L < \varepsilon$$

Now, let $\delta = \min\{\delta_1, \delta_2\}$, then for all $x \in E$ with $0 < |x - p| < \delta$

$$-\varepsilon < g(x) - L \le f(x) - L \le h(x) - L < \varepsilon$$

So for all $x \in E$ with $0 < |x - p| < \delta$

$$-\varepsilon < f(x) - L < \varepsilon$$

The last double inequality is equivalent to $|f(x) - L| < \varepsilon$, which completes the proof.

14. Let f, g be real-valued functions defined on $E \subset \mathbb{R}$ and let p be a limit point of E.

a, If $\lim_{x\to p} f(x)$ and $\lim_{x\to p} (f(x) + g(x))$ exist, prove that $\lim_{x\to p} g(x)$ exists.
Let $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} (f(x) + g(x)) = C$. Then by Theorem 4.1.6, Part b, $\lim_{x\to p} (-f(x)) = -A$ and by Part a of the same theorem

$$\lim_{x\to p} g(x) = \lim_{x\to p} [(f(x) + g(x)) + (-f(x))] = C - A$$

b. If $\lim_{x\to p} f(x)$ and $\lim_{x\to p} (f(x)\, g(x))$ exist, does it follow that $\lim_{x\to p} g(x)$ exists?
The answer is no. Take for instance $E = \mathbb{R} \setminus \{0\}$ and $f : E \to \mathbb{R}$, $f(x) = x^2$ and $f : E \to \mathbb{R}$, $g(x) = \frac{1}{x}$. Then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$$
$$\lim_{x\to 0} (f(x)g(x)) = \lim_{x\to 0} x = 0$$

but $\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{1}{x}$ is undefined.

17. Investigate the limits at ∞ of each of the following functions defined on $(0, \infty)$.

a. $f(x) = \frac{3x^2+3x-1}{2x^2+1}$
Observe that for all $x \in (0, \infty)$

$$f(x) = \frac{3x^2 + 3x - 1}{2x^2 + 1} = \frac{3 + \frac{3}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}}$$

Then by Theorem 4.1.6 Parts a and c, and the remark on page 127:
"Readers should convince themselves that all theorems up to this point involving limits at a point $p \in \mathbb{R}$ are still valid if p is replaced by ∞ or $-\infty$."

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3+\frac{3}{x}-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{3}{2}$$

d. $f(x) = \frac{2x+3}{\sqrt{x+1}}$
We will use the sequential theorem for limits and the remark on page 127:
"Readers should convince themselves that all theorems up to this point involving limits at a point $p \in \mathbb{R}$ are still valid if p is replaced by ∞ or $-\infty$."
Let $p_n = n$. Then for all $n \in \mathbb{N}$

$$\begin{aligned} f(p_n) &= f(n) = \frac{2n+3}{\sqrt{n+1}} \geq \frac{2n}{\sqrt{n+1}} \geq \frac{2n}{\sqrt{2n+(1-n)}} \\ &\geq \frac{2n}{\sqrt{2n}} = \sqrt{2n} \end{aligned}$$

Hence, the sequence $\{f(p_n)\}$ is unbounded, so $\lim_{x\to\infty} f(x)$ does not exist.

h. $f(x) = x \sin \frac{1}{x}$
Note that $f : (0,\infty) \to \mathbb{R}$. Therefore, if $x = \frac{1}{y}$ is used as an argument of f, then $y > 0$ so $y \to 0$ if and only if $x \to \infty$ and

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0} f\left(\frac{1}{y}\right) = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

4.2 Continuous Functions

1. For each of the following, determine whether the given function is continuous at the indicated point x_0.

a. $f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$ at $x_0 = 3$.
Observe $f : \mathbb{R} \to \mathbb{R}$, so 3 is a limit point of the domain of f which means that f is continuous at 3 if and only if

$$\lim_{x\to 3} f(x) = f(3)$$

Moreover, if $x \neq 3$

$$\frac{2x^2-5x-3}{x-3} = \frac{(2x+1)(x-3)}{x-3} = 2x+1$$

therefore

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{2x^2-5x-3}{x-3} = \lim_{x\to 3}(2x+1) = 7 \neq 6 = f(3)$$

We conclude that the function f is not continuous at 3.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 8x & \text{when } x \text{ is rational} \\ 2x^2+8 & \text{when } x \text{ is irrational} \end{cases}$$

a. Prove, using ε and δ, that f is continuous at 2.
Let $\varepsilon > 0$.

- Consider

$$|8x - f(2)| = |8x - 16| = 8|x-2|$$

Let $\delta_1 = \frac{\varepsilon}{8}$. Then for all x with $|x-2| < \delta_1$

$$|8x - f(2)| = 8|x-2| < 8\delta_1 = 8 \cdot \frac{\varepsilon}{8} = \varepsilon$$

- Consider

$$\left|2x^2+8-f(2)\right| = \left|2x^2-8\right| = 2|x+2||x-2| \leq 2(|x|+2)|x-2|$$

Let $\delta_2 \leq 1$. Then for all x with $|x-2| < \delta_2$

$$|x| - 2 \leq |x-2| < \delta_2 \leq 1$$

and

$$\left|2x^2+8-f(2)\right| \leq 2(|x|+2)|x-2| \leq 10|x-2|$$

Choose $\delta_2 = \min\left\{1, \frac{\varepsilon}{10}\right\}$. Then for all x with $|x-2| < \delta_2$

$$\left|2x^2+8-f(2)\right| \leq 10|x-2| < 10\delta_2 \leq 10 \cdot \frac{\varepsilon}{10} = \varepsilon$$

We conclude that if $\delta = \min\{\delta_1, \delta_2\} = \min\left\{1, \frac{\varepsilon}{10}\right\}$

$$|f(x) - f(2)| < \varepsilon$$

whenever $|x-2| < \delta$. This implies that f is continuous at 2.

b. Is f continuous at 1? Justify your answer.

The answer is no. Observe that $f(1) = 8$. Since 1 is a limit point of $\mathbb{R}$, f is continuous at 1 if and only if $\lim_{x\to 1} f(x) = f(1) = 8$. We will use the sequential criterion for limits to show that even if $\lim_{x\to 1} f(x)$ exists, it does not equal 8.

Choose $i_n \notin \mathbb{Q}$ such that $1 < i_n < 1 + \frac{1}{n}$. Then $i_n \to 1$ as $n \to \infty$, $i_n \neq 1$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} f(i_n) = \lim_{n\to\infty} \left(2i_n^2 + 8\right) = 10$$

According to the sequential theorem for limits this implies that if $\lim_{x\to 1} f(x)$ exists, then $\lim_{x\to 1} f(x) = 10 \neq 8 = f(1)$. We conclude that f is not continuous at 1. Note that in actuality $\lim_{x\to 1} f(x)$ does not exist, but we did not need that fact to prove that f is not continuous at 1.

5. Define $f : (0,1] \to \mathbb{R}$ by

$f(x) = \frac{1}{\sqrt{x}} - \sqrt{\frac{x+1}{x}}$.

a. Justify that f is continuous on $(0,1]$.

The result of Exercise 7b, Page 129, guarantees that $\sqrt{x}$ and $\sqrt{x+1}$ are continuous on $(0,1]$ (see the key). Then Theorem 4.2.3, Part c, guarantees that $\frac{1}{\sqrt{x}}$ and $\frac{\sqrt{x+1}}{\sqrt{x}}$ are continuous on $(0,1]$. Finally, Theorem 4.2.3, Part a, guarantees that f is continuous on $(0,1]$.

b. Can one define $f(0)$ so that f is continuous on $[0,1]$?

The answer is yes. To prove this, we will show that $\lim_{x\to 0} f(x) = 0$. Let $\varepsilon > 0$. Consider

$$\begin{aligned} |f(x) - 0| &= \left|\frac{1}{\sqrt{x}} - \sqrt{\frac{x+1}{x}}\right| = \left|\frac{1-\sqrt{x+1}}{\sqrt{x}}\right| = \left|\frac{1-(x+1)}{\sqrt{x}\left(1+\sqrt{x+1}\right)}\right| \\ &= \left|\frac{-\sqrt{x}}{1+\sqrt{x+1}}\right| < \sqrt{x} \end{aligned}$$

Choose $\delta = \varepsilon^2$. Then for all $x \in (0,1]$ with $(0 <)\ |x| < \delta = \varepsilon^2$

$$|f(x) - 0| < \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

We conclude that the function

$$g(x) = \begin{cases} f(x) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

is continuous on $[0,1]$.

8. a. Prove that $f(x) = \cos x$ is continuous on $\mathbb{R}$.

Let $p \in \mathbb{R}$ and $\varepsilon > 0$. Consider

$$\begin{aligned} |f(x) - f(p)| &= |\cos x - \cos p| = \left| -2\sin\left[\frac{1}{2}(x-p)\right]\sin\left[\frac{1}{2}(x+p)\right]\right| \\ &\leq 2\left|\sin\left[\frac{1}{2}(x-p)\right]\right| \leq 2\left|\frac{1}{2}(x-p)\right| = |x-p| \end{aligned}$$

Choose $\delta = \varepsilon$, then for all x with $|x-p| < \delta = \varepsilon$

$$|f(x) - f(p)| \leq |x-p| < \delta = \varepsilon$$

This completes the proof.

b. If $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous on E, prove that $g(x) = \cos(f(x))$ is continuous on E.

Work from the outside in.

- Let $p \in E$ and $\varepsilon > 0$. Then, because $\cos x$ is continuous on $\mathbb{R}$, there exists a $\delta_1 > 0$, such that
$$|\cos u - \cos(f(p))| < \varepsilon$$
whenever $|u - f(p)| < \delta_1$.
- Similarly, because f is continuous on E, there exists a $\delta > 0$, such that
$$|f(x) - f(p)| < \delta_1$$
whenever $|x-p| < \delta$.

We conclude that

$$|g(x) - g(p)| = |\cos(f(x)) - \cos(f(p))| < \varepsilon$$

whenever $|x-p| < \delta$. This shows that g is continuous at p. Since p was chosen arbitrarily in E, we may conclude that g is continuous on E.