Chapter 1

The Real Numbers

1.1 The Real Number System

1. Write the following expressions in equivalent forms not involving absolute values.

   b. $a + b - |a - b|$

   Solution

   In order to eliminate the absolute value we distinguish two cases, $a \geq b$ and $a < b$.

   - Case 1: $a \geq b$
     Then $a + b - |a - b| = a + b - (a - b) = 2b = 2\min\{a, b\}$
   - Case 2: $a < b$
     Then $a + b - |a - b| = a + b + (a - b) = 2a = 2\min\{a, b\}$

   We conclude that for all real numbers a and b, $a + b - |a - b| = 2\min\{a, b\}$.

2. Let F denote the field consisting of the set $\{0, 1\}$ with operations $+$ and $\cdot$ defined as

$$0 + 0 = 1 + 1 = 0, 1 + 0 = 0 + 1 = 1$$
$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1$$

   Show that it is impossible to define an order $<$ on the field F that has properties

   (F) If $a, b \in F$, then exactly one of the following is true:

$$a < b, a = b, \text{ or } b < a.$$

   (G) If $a, b, c \in F$, and $a < b$, and $b < c$, then $a < c$.

   (H) If $a, b, c \in F$, and $a < b$ then $a + c < b + c$, and if $0 < c$, then $ac < bc$.

Solution

In class I gave you the hint that the first part of property (H) would break. We follow up on this suggestion. Observe that for this field 0 is the additive identity, and 1 is the multiplicative identity, so $0 \neq 1$. According to property (F), we are left with two possibilities, either $0 < 1$, or $1 < 0$. We will show that both lead to a contradiction.

- Case 1: $0 < 1$
  Then, by the first part of property (H), $0+1 < 1+1$, which implies $1 < 0$, a contradiction.
- Case 2: $1 < 0$
  Then, again by the first part of property (H), $1 + 1 < 0 + 1$, which implies $0 < 1$, a contradiction.

This completes the proof.

4. Show that $\sqrt{p}$ is irrational if p is prime.

   Solution

   Use a proof by contradiction. Suppose $m, n \in \mathbb{Z}$ and $\sqrt{p} = \frac{m}{n}$ in its most reduced form. Then $pn^2 = m^2$, so p divides m^2, hence p divides m. Let $m = kp$, then $pn^2 = (kp)^2$, so $n^2 = k^2 p$. We conclude that p divides n^2. Therefore p divides n, a contradiction.

5. Find the supremum and infimum of each S. State whether they are in S.

   a. $S = \left\{x \mid x = -\left(1/n\right) + \left[1 + (-1)^n\right] n^2, n \in \mathbb{Z}^+\right\}$
   Solution
   First, we evaluate the elements of S corresponding to $n = 1, 2, \cdots, 10$ and compute their decimal approximations

   $$-1, 15/2, -1/3, 127/4, -1/5, 431/6, -1/7, 1023/8, -1/9, 1999/10$$
   $$-1., 7.500, -.3333, 31.75, -.2000, 71.83, -.1429, 127.9, -.1111, 199.9$$

   Notice that the term $\left[1 + (-1)^n\right] n^2$ vanishes for odd values of n. Therefore, if n is odd $x = -\left(1/n\right)$, which creeps up slowly to zero. If n is even, the n^2 term will make the expression $-\left(1/n\right) + \left[1 + (-1)^n\right] n^2$ increase rapidly. We conclude that $\inf S = -1$. The set has no upper bound. For convenience we write $\sup S = \infty$. Note that $-1 \in S$, while $\infty \notin S$.

   b. $S = \left\{x \mid x^2 < 9\right\}$
   Solution
   We solve the inequality $x^2 < 9$ for x, and conclude that $-3 < x < 3$. Hence $\inf S = -3$ and $\sup S = 3$, neither of which belong to S.

d. $S = \{x \mid |2x+1| < 5\}$
Solution
Solve the inequality $|2x+1| < 5$, for x by rewriting it as

$$-5 < 2x + 1 < 5$$

Elementary arithmetic shows that $-3 < x < 2$, so $\inf S = -3$ and $\sup S = 2$, neither of which belong to S.

e. $S = \left\{x \mid \left(x^2+1\right)^{-1} > 1/2\right\}$
Solution
We rewrite the inequality $\left(x^2+1\right)^{-1} > 1/2$ as

$$x^2 + 1 < 2$$

and discover that $-1 < x < 1$. Therefore $\inf S = -1$ and $\sup S = 1$, neither of which belong to S.

7. a. Show that

$$\inf S \leq \sup S$$

for any nonempty set of real numbers, and give necessary and sufficient conditions for equality.
Solution
First, let us assume that S is bounded, then $\inf S$ and $\sup S$ are both real numbers. Since $S \neq \phi$ there exist an $x \in S$ and

$$\inf S \leq x \leq \sup S$$

which implies the desired result.
Next, we consider the case that S is unbounded above but still bounded below. Again, since $S \neq \phi$ there exist an $x \in S$ and

$$\inf S \leq x < \infty$$

($x < \infty$, because x is real). Finally, because S is unbounded above, we decided to write $\sup S = \infty$. Therefore

$$\inf S \leq x < \infty = \sup S$$

Hence, in spite of the unboundedness of S, we may still write $\inf S \leq \sup S$. The remaining two cases can be proved in a similar fashion.
Finally, we conjecture that $\inf S = \sup S$ if and only if S has exactly one element.

- Proof of the **if** part.
  Suppose $S = \{x\}$, then $\inf S = x$ and $\sup S = x$, so $\inf S = \sup S$.
- Proof of the **only if** part.
  Suppose $\inf S = \sup S = \gamma$. Then γ must be real. Because $S \neq \phi$ there exist an $x \in S$ and

$$\gamma = \inf S \leq x \leq \sup S = \gamma$$

  We conclude that γ is the only element of S.

10. a. Let S and T be nonempty sets of real numbers and define

$$S + T = \{s + t \mid s \in S, t \in T\}$$

Show that

$$\sup(S + T) = \sup S + \sup T$$

Solution

We will show that $\sup(S+T) \leq \sup S + \sup T$ and $\sup(S+T) \geq \sup S + \sup T$.

To prove the first inequality, we let $u \in S + T$. Then there exist $s \in S$ and $t \in T$ such that $u = s + t$, and

$$u = s + t \leq \sup S + \sup T$$

So, $\sup S + \sup T$ is an upper bound for $S + T$ and we may conclude that $\sup(S+T) \leq \sup S + \sup T$.

To prove the second inequality, we choose $s \in S$ and $t \in T$. Then $s + t \in S + T$, therefore

$$s + t \leq \sup(S+T) \text{ , hence } s \leq \sup(S+T) - t$$

Now think of $t \in T$ as being arbitrary but fixed. Then $\sup(S+T) - t$ is an upper bound of S, so

$$\sup S \leq \sup(S+T) - t, \text{ hence } t \leq \sup(S+T) - \sup S$$

These inequalities hold true for all $t \in T$, therefore $\sup(S+T) - \sup S$ is an upper bound of T, so

$$\sup T \leq \sup(S+T) - \sup S$$

which implies $\sup(S+T) \geq \sup S + \sup T$. This completes the proof.

1.2 Mathematical Induction

3. Prove by induction.

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n\left(4n^2-1\right)}{3}$$

Solution

Let P_n denote the proposition

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n\left(4n^2-1\right)}{3}$$

- We verify that P_1 is true.

$$\sum_{k=1}^{1} (2k-1)^2 = 1^2 = 1 \text{ and } \frac{1\,(4-1)}{3} = \frac{3}{3} = 1$$

- Assume that P_n is true for some $n \in \mathbb{N}$, then

$$\begin{aligned}
\sum_{k=1}^{n+1} (2k-1)^2 &= \sum_{k=1}^{n} (2k-1)^2 + (2\,(n+1)-1)^2 \\
&= \frac{n\left(4n^2-1\right)}{3} + (2n+1)^2 = \frac{n\,(2n-1)\,(2n+1)}{3} + (2n+1)^2 \\
&= \frac{1}{3}\,(n\,(2n-1) + 3\,(2n+1))\,(2n+1) = \frac{1}{3}\left(2n^2+5n+3\right)(2n+1) \\
&= \frac{2}{3}\,(n+1)\left(n+\frac{3}{2}\right)(2n+1) = \frac{1}{3}\,(n+1)\,((2n+3)\,(2n+1)) \\
&= \frac{1}{3}\,(n+1)\left(4n^2+8n+3\right) = \frac{1}{3}\,(n+1)\left(4\,(n+1)^2-1\right)
\end{aligned}$$

This shows that P_{n+1} is true. Hence, by the principle of mathematical induction, we may conclude that P_n is true for all $n \in \mathbb{N}$.

15. Let $a_1 = a_2 = 5$ and

$$a_{n+1} = a_n + 6a_{n-1}, \quad n \geq 2$$

Show by induction that $a_n = 3^n - (-2)^n$ if $n \geq 1$.

Solution

Let P_n denote the proposition

$$a_n = 3^n - (-2)^n$$

.

- We verify that P_1 and P_2 are true. Observe that

$$3^1 - (-2)^1 = 3 + 2 = 5 \text{ and } 3^2 - (-2)^2 = 9 - 4 = 5$$

- Assume that for some integer $n \geq 2$, $P_1, P_2, \cdots, P_n$ are all true. Then

$$\begin{aligned} a_{n+1} &= a_n + 6a_{n-1} = 3^n - (-2)^n + 6\left(3^{n-1} - (-2)^{n-1}\right) \\ &= \left(1 + \frac{6}{3}\right)3^n + \left(-1 - \frac{6}{-2}\right)(-2)^n = 3 \cdot 3^n + 2 \cdot (-2)^n = 3^{n+1} - (-2)^{n+1} \end{aligned}$$

  This shows that P_{n+1} is true.

Hence, by the principle of mathematical induction, we may conclude that P_n is true for all $n \geq 1$.

18. Prove by induction that

$$\int_0^1 y^n (1-y)^r \, dy = \frac{n!}{(r+1)(r+2)\cdots(r+n+1)}$$

if n is a nonnegative integer and $r > -1$.

Solution

Let P_n denote the stated proposition.

- We verify that P_0 is true.

$$\int_0^1 (1-y)^r \, dy = -\left[\frac{(1-y)^{r+1}}{r+1}\right]_0^1 = -\left(0 - \frac{1}{r+1}\right) = \frac{1}{r+1} = \frac{0!}{r+1}$$

- Assume that P_n is true for some nonnegative integer n. As I mentioned in class, you should use integration by parts to show this implies that P_{n+1} is true. In order to be able to use the induction assumption we differentiate the y^{n+1} term

$$\int_0^1 y^{n+1} (1-y)^r \, dy = \left[-\frac{y^{n+1}(1-y)^{r+1}}{r+1}\right]_0^1 + \frac{n+1}{r+1}\int_0^1 y^n (1-y)^{r+1} \, dy$$

  Observe the integrated term equals zero. Moreover, since $r > -1$ certainly $r + 1 > -1$, so we may apply the induction assumption to the integral $\int_0^1 y^n (1-y)^{r+1} \, dy$, which yields

$$\begin{aligned} \int_0^1 y^{n+1} (1-y)^r \, dy &= \frac{n+1}{r+1} \cdot \frac{n!}{(r+2)(r+3)\cdots(r+n+2)} \\ &= \frac{(n+1)!}{(r+1)(r+2)\cdots(r+(n+1)+1)} \end{aligned}$$

This shows that P_{n+1} is true.

Hence, by the principle of mathematical induction, we may conclude that P_n is true for all nonnegative integers n.

1.3 The Real Line

1. Find $S \cap T, (S \cap T)^c, S^c \cap T^c, S \cup T, (S \cup T)^c$, and $S^c \cup T^c$.

   a. $S = (0, 1), T = \left[\frac{1}{2}, \frac{3}{2}\right]$
      - $S \cap T = \left[\frac{1}{2}, 1\right)$
      - $(S \cap T)^c = \left(-\infty, \frac{1}{2}\right) \cup [1, \infty)$
      - $S^c \cap T^c = (S \cup T)^c = \left(\left(0, \frac{3}{2}\right]\right)^c = (-\infty, 0] \cup \left(\frac{3}{2}, \infty\right)$
      - $S \cup T = \left(0, \frac{3}{2}\right]$
      - $(S \cup T)^c = S^c \cap T^c = (-\infty, 0] \cup \left(\frac{3}{2}, \infty\right)$
      - $S^c \cup T^c = (S \cap T)^c = \left(-\infty, \frac{1}{2}\right) \cup [1, \infty)$ Note: The answer in the back of the book is not correct.

2. Let $S_k = \left(1 - \frac{1}{k}, 2 + \frac{1}{k}\right], k \geq 1$. Find

   a. $\cup_{k=1}^{\infty} S_k = (0, 3]$
   b. $\cap_{k=1}^{\infty} S_k = [1, 2]$ Note: The answer in the back of the book is not correct.
   c. $\cup_{k=1}^{\infty} S_k^c = (\cap_{k=1}^{\infty} S_k)^c = ([1, 2])^c = (-\infty, 1) \cup (2, \infty)$
   d. $\cap_{k=1}^{\infty} S_k^c = (\cup_{k=1}^{\infty} S_k)^c = ((0, 3])^c = (-\infty, 0] \cup (3, \infty)$

3. Prove: If A and B are sets and there is a set X such that $A \cup X = B \cup X$ and $A \cap X = B \cap X$, then $A = B$.

   Solution

   In class I made the remark that you should prove that $A \cup X = B \cup X$ implies $A - X = B - X$.

   We show that $A - X \subset B - X$. Let $s \in A - X$. Then $s \in A$ and $s \notin X$, so $s \in A \cup X$ and $s \notin X$. Since $A \cup X = B \cup X$ this implies that $s \in B \cup X$ and $s \notin X$. Therefore $s \in B - X$, hence $A - X \subset B - X$. In a similar fashion we can show that $B - X \subset A - X$. We may then conclude that $A - X = B - X$. Finally

   $$A = (A \cap X) \cup (A - X) = (B \cap X) \cup (B - X) = B$$

4. Find the largest ϵ such that S contains an ϵ-neighborhood of x_0.

b. $x_0 = \frac{2}{3}, S = \left[\frac{1}{2}, \frac{3}{2}\right]$

Solution

$\epsilon = \min\left\{\frac{2}{3} - \frac{1}{2}, \frac{3}{2} - \frac{2}{3}\right\} = \frac{1}{6}$

7. Let $\mathcal{F}$ be a collection of sets and define

$$I = \cap\{F \mid F \in \mathcal{F}\}$$

a. Prove that $I^c = \cup\{F^c \mid F \in \mathcal{F}\}$

Solution

- Part 1: We show that $I^c \subset \cup\{F^c \mid F \in \mathcal{F}\}$
  Let $s \in I^c$, then $s \notin I$. Hence, there exists an $F_s \in \mathcal{F}$ such that $s \notin F_s$. Then $s \in F_s^c \subset \cup\{F^c \mid F \in \mathcal{F}\}$. We conclude that $I^c \subset \cup\{F^c \mid F \in \mathcal{F}\}$.
- Part 2: We show that $\cup\{F^c \mid F \in \mathcal{F}\} \subset I^c$
  Let $s \in \cup\{F^c \mid F \in \mathcal{F}\}$. Hence, there exists an $F_s \in \mathcal{F}$ such that $s \in F_s^c$. Then $s \notin F_s$, so $s \notin \cap\{F \mid F \in \mathcal{F}\} = I$. We conclude that $s \in I^c$ and therefore $\cup\{F^c \mid F \in \mathcal{F}\} \subset I^c$.

Together, Part 1 and Part 2 establish the desired result.

8. a. Show that the intersection of finitely many open sets is open.

Solution

For $k, n \in \mathbb{N}, k \leq n$, let S_k denote an open set. We will show that the intersection

$$I = \cap_{k=1}^{n} S_k$$

is open.

Let $s \in I$, then $s \in S_k$ for $k = 1, 2, \cdots, n$. Since each set S_k is open, there exist $\epsilon_k > 0$ such that the ϵ_k-neighborhood $(s - \epsilon_k, s + \epsilon_k)$ is contained in S_k. We define

$$\epsilon = \min_{1 \leq k \leq n} \epsilon_k$$

then

$$(s - \epsilon, s + \epsilon) \subset S_k \text{ for } 1 \leq k \leq n$$

hence

$$(s - \epsilon, s + \epsilon) \subset \cap_{k=1}^{n} S_k = I$$

so s is an interior point of I, and because s was chosen arbitrarily in I, this means that I is open.

b. Give an example showing that the intersection of infinitely many open sets may fail to be open.

Solution

Observe

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$$

15. Prove or disprove: A set has no limit points if and only if each of its points is isolated.

Solution

This statement is not correct. In our discussion of boundary points, we investigated the set

$$S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$$

The same set also provides a perfect counter example for the proposition above. Observe that every point of S is an isolated point, yet zero is a limit point of S.

16. a. Prove: If S is bounded above and $\beta = \sup S$, then $\beta \in \partial S$.

Solution

We must show that every ϵ-neighborhood N of β contains a point in S and a point not in S. Let $\epsilon > 0$ and $N = (\beta - \epsilon, \beta + \epsilon)$. Then, since $\beta = \sup S$, there exists a number $x_1 \in S$ with $\beta - \epsilon < x_1 \leq \beta$. Moreover, since β is an upper bound of S, the number $x_2 = \beta + \frac{1}{2}\epsilon$ is not in S. We conclude that β is a boundary point of S.

b. State the analogous result for a set bounded below.

Solution

If S is bounded below and $\alpha = \inf S$, then $\alpha \in \partial S$.

17. Prove: If S is closed and bounded, then $\inf S$ and $\sup S$ are both in S.

Solution

Let $\alpha = \inf S$ and $\beta = \sup S$. In class I suggested you use the fact that S is closed if and only if $S = \overline{S}$. We will show that $\beta \in S$. The argument goes like this. Since S is closed, $S = \overline{S}$, so $S = S \cup \partial S$. This means that ∂S is a subset of S. Moreover, by Exercise 16 Part a, $\beta \in \partial S$. Hence, $\beta \in S$. In a similar manner we can prove that $\alpha \in S$.