Chapter 2

Differential Calculus of Functions of One Variable

2.1 Functions and Limits

1. Each of the following conditions fails to define a function on any domain. State why.

   a. $\sin f(x) = x$

   Solution

   - If $|x| > 1$ this equation has no (real) solution for $f(x)$.
   - If $|x| \leq 1$, each of the values

   $$f(x) = \arcsin x + 2n\pi, n \in \mathbb{Z} \quad \text{and} \quad f(x) = \pi - \arcsin x + 2n\pi, n \in \mathbb{Z}$$

   will satisfy the given equation equation. Hence the assigned value $f(x)$ is not unique and therefore f is not a function.

   b. $e^{f(x)} = -|x|$

   Solution

   Since the exponential function $g(x) = e^x$ is strictly positive, the given equation has no solution for any value of x.

   c. $1 + x^2 + [f(x)]^2 = 0$

   Solution

   Since $1 + x^2 + [f(x)]^2 \geq 1$, the given equation has no solution for any value of x.

   d. $f(x)[f(x) - 1] = x^2$

   Solution

This equation is equivalent to $f^2(x) - f(x) - x^2 = 0$. Using the quadratic formula, we obtain

$$f(x) = \frac{1 \pm \sqrt{1+4x^2}}{2}$$

Again, the assigned value $f(x)$ is not unique and therefore f is not a function.

3. Find D_f.

   a. $f(x) = \tan x$
   Solution
   $D_f = \mathbb{R} - \left\{\frac{\pi}{2} + n\pi \mid n \in \mathbb{Z}\right\}$

   d. $f(x) = \frac{\sin x}{x}$
   Solution
   $D_f = \mathbb{R} - \{0\}$

   e. $e^{[f(x)]^2} = x, f(x) \geq 0$
   Solution
   Since e^{y^2} attains all values greater than or equal to one and none less than one, the given equation has a solution if and only if $x \geq 1$. Hence $D_f = [1, \infty)$. (of course this is commensurate with the equivalent equation $f(x) = \sqrt{\ln x}$)

4. Find $\lim_{x \to x_0} f(x)$ and justify your answers with an ϵ-δ proof.

   a. $\lim_{x \to 1} \left(x^2 + 2x + 1\right) = 4$
   Solution
   Let $f(x) = x^2 + 2x + 1, L = 4$, and $\epsilon > 0$. Consider

$$|f(x) - L| = \left|\left(x^2 + 2x + 1\right) - 4\right| = \left|x^2 + 2x - 3\right| = |x+3|\,|x-1|$$

   Let $\delta \leq 1$. Then for all x with $0 < |x-1| < \delta \leq 1$

$$-1 \leq x - 1 \leq 1, \text{ so } \quad 3 \leq x + 3 \leq 5$$

   Hence,

$$|f(x) - L| = |x+3|\,|x-1| \leq 5\,|x-1|$$

   Choose $\delta = \min\left\{1, \frac{\epsilon}{5}\right\}$, then for all x with $0 < |x-1| < \delta$

$$|f(x) - L| \leq 5\,|x-1| < 5 \cdot \frac{\epsilon}{5} = \epsilon$$

   This completes the proof.

b. $\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)\left(x^2+2x+4\right)}{x-2}=\lim_{x\to 2}\left(x^2+2x+4\right)=12$

Solution

Let $f(x)=\frac{x^3-8}{x-2}, L=12$, and $\epsilon>0$. For $x\neq 2$

$$|f(x)-L|=\left|\frac{x^3-8}{x-2}-12\right|=\left|x^2+2x-8\right|=|x+4|\,|x-2|$$

Let $\delta\leq 1$. Then for all x with $0<|x-2|<\delta\leq 1$

$$-1\leq x-2\leq 1,\text{ so}\quad 5\leq x+4\leq 7$$

Hence,

$$|f(x)-L|=|x+4|\,|x-2|\leq 7\,|x-2|$$

Choose $\delta=\min\left\{1,\frac{\epsilon}{7}\right\}$, then for all x with $0<|x-2|<\delta$

$$|f(x)-L|\leq 7\,|x-2|<7\cdot\frac{\epsilon}{7}=\epsilon$$

This completes the proof.

d. $\lim_{x\to 4}\sqrt{x}=2$

Solution

Let $f(x)=\sqrt{x}, L=2$, and $\epsilon>0$. Consider

$$|f(x)-L|=\left|\sqrt{x}-2\right|=\left|\frac{x-4}{\sqrt{x}+2}\right|=\frac{1}{\sqrt{x}+2}\,|x-4|\leq\frac{1}{2}\,|x-4|$$

Choose $\delta=2\epsilon$, then for all $x\in D_f$ with $0<|x-4|<\delta$

$$|f(x)-L|\leq\frac{1}{2}\,|x-4|<\frac{1}{2}\cdot 2\epsilon=\epsilon$$

This completes the proof.

7. Find $\lim_{x\to x_0-}f(x)$ and $\lim_{x\to x_0+}f(x)$, if they exist. Use ϵ-δ proofs, where applicable, to justify your answers.

b. $x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|},\qquad x_0=0$

Solution

- $\lim_{x\to 0-}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|}\right)$
  Observe that

$$\begin{aligned}\lim_{x\to 0-}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|}\right) &= \lim_{x\to 0-}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\left(-\frac{1}{x}\right)\right)\\ &= \lim_{x\to 0-}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}-\sin\frac{1}{x}\right)\\ &= \lim_{x\to 0-} x\cos\frac{1}{x}\end{aligned}$$

  Finally, we will now show that $\lim_{x\to 0-} x\cos\frac{1}{x}=0$. Let $\epsilon>0$. Consider

$$\left|x\cos\frac{1}{x}-0\right| = |x|\left|\cos\frac{1}{x}\right| \le |x|$$

  Choose $\delta=\epsilon$, then for all x with $-\delta<x<0$

$$\left|x\cos\frac{1}{x}-0\right| \le |x| < \delta = \epsilon$$

  We conclude that

$$\lim_{x\to 0-}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|}\right) = \lim_{x\to 0-} x\cos\frac{1}{x} = 0$$

- $\lim_{x\to 0+}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|}\right)$
  Above we proved that $\lim_{x\to 0-} x\cos\frac{1}{x}=0$. In a similar manner it can be shown that $\lim_{x\to 0+} x\cos\frac{1}{x}=0$, therefore

$$\begin{aligned}\lim_{x\to 0+}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{|x|}\right) &= \lim_{x\to 0+}\left(x\cos\frac{1}{x}+\sin\frac{1}{x}+\sin\frac{1}{x}\right)\\ &= \lim_{x\to 0+}\left(x\cos\frac{1}{x}+2\sin\frac{1}{x}\right)\\ &= \lim_{x\to 0+} x\cos\frac{1}{x}+\lim_{x\to 0+} 2\sin\frac{1}{x}\\ &= \lim_{x\to 0+} 2\sin\frac{1}{x}\end{aligned}$$

  We now prove that $\lim_{x\to 0+} 2\sin\frac{1}{x}$ does not exist. Let $L\in\mathbb{R}$, $\delta>0$ and $\epsilon_0 = \max\{|2-L|,|2+L|\}$. By the Archimedean property of $\mathbb{R}$, $\exists n\in\mathbb{N}$ such that

$$0<x_1=\frac{1}{\frac{\pi}{2}+2n\pi}<\delta \quad\text{and}\quad 0<x_2=\frac{1}{\frac{3\pi}{2}+2n\pi}<\delta$$

Note that $\left|2\sin\frac{1}{x_1} - L\right| = |2-L|$ and $\left|2\sin\frac{1}{x_2} - L\right| = |-2-L| = |2+L|$. Hence, for every $L \in \mathbb{R}$, there exists an $\epsilon_0 > 0$ such that for every $\delta > 0$, there exists an x with $0 < x < \delta$, with $\left|2\sin\frac{1}{x} - L\right| \geq \epsilon_0$. This shows that $\lim_{x\to 0+} 2\sin\frac{1}{x}$ does not exist. Therefore

$$\lim_{x\to 0+}\left(x\cos\frac{1}{x} + \sin\frac{1}{x} + \sin\frac{1}{|x|}\right)$$

does not exist.

c. $\frac{|x-1|}{x^2+x-2}, \quad x_0 = 1$

Solution

Use Theorem 2.1.4 adapted for one-sided limits.

- $\lim_{x\to 1-}\frac{|x-1|}{x^2+x-2} = \lim_{x\to 1-}\left(-\frac{x-1}{x^2+x-2}\right) = \lim_{x\to 1-}\left(-\frac{1}{x+2}\right) = -\frac{1}{3}$
- $\lim_{x\to 1+}\frac{|x-1|}{x^2+x-2} = \lim_{x\to 1+}\frac{x-1}{x^2+x-2} = \lim_{x\to 1+}\frac{1}{x+2} = \frac{1}{3}$

8. Prove: If $h(x) \geq 0$ for $a < x < x_0$ and $\lim_{x\to x_0-} h(x)$ exists, then $\lim_{x\to x_0-} h(x) \geq 0$. Conclude from this that if $f_2(x) \geq f_1(x)$ for $a < x < x_0$, then

$$\lim_{x\to x_0-} f_2(x) \geq \lim_{x\to x_0-} f_1(x)$$

if both limits exist.

Solution

In class I suggested you use a proof by contradiction. The essential ingredient of the proof can be found in the proof of Theorem 2.1.4. Part 4.

Assume $\lim_{x\to x_0-} h(x) = L < 0$. Then $\exists\delta > 0$ such that for all x with $x_0 - \delta < x < x_0$

$$|h(x) - L| < \frac{|L|}{2}$$

This implies that

$$-\frac{|L|}{2} < h(x) - L < \frac{|L|}{2}$$

so

$$-\frac{|L|}{2} < h(x) + |L| < \frac{|L|}{2} \quad \text{and thus} \quad -\frac{3|L|}{2} < h(x) < -\frac{|L|}{2}$$

a contradiction with the fact that $h(x) \geq 0$ for $a < x < x_0$.

Finally, choosing $h(x) = f_2(x) - f_1(x)$ yields

$$\lim_{x\to x_0-}(f_2(x) - f_1(x)) \geq 0 \quad \text{and thus} \quad \lim_{x\to x_0-} f_2(x) \geq \lim_{x\to x_0-} f_1(x)$$

Provided both limits exist.

15. Find $\lim_{x\to\infty} f(x)$ if it exists, and justify your answer directly from Definition 2.1.7.

   b. $\frac{\sin x}{x^\alpha} \quad (\alpha > 0)$

   Solution

   We will show that $\lim_{x\to\infty} \frac{\sin x}{x^\alpha} = 0$. Let $\epsilon > 0$. Observe that

$$\left|\frac{\sin x}{x^\alpha} - 0\right| = \frac{|\sin x|}{|x|^\alpha} \le \frac{1}{|x|^\alpha}$$

   Choose $\tau = (1/\epsilon)^{1/\alpha}$ then for all $x > \tau$

$$\left|\frac{\sin x}{x^\alpha} - 0\right| \le \frac{1}{|x|^\alpha} < \frac{1}{\tau^\alpha} = \frac{1}{\left((1/\epsilon)^{1/\alpha}\right)^\alpha} = \epsilon$$

   This completes the proof.

   f. $e^{-x^2}e^{2x}$

   Solution

   We will show that $\lim_{x\to\infty}\left(e^{-x^2}e^{2x}\right) = 0$. Let $\epsilon > 0$. Observe that for $x > 4$

$$\left|e^{-x^2}e^{2x} - 0\right| = e^{-x^2+2x} = e^{-\frac{1}{2}x^2+2x}e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x(x-4)}e^{-\frac{1}{2}x^2} < e^{-\frac{1}{2}x^2}$$

   - Note that if $\epsilon > 1$, then $e^{-\frac{1}{2}x^2} < \epsilon$. Hence, with $\tau = 4$ and $x > \tau$

$$\left|e^{-x^2}e^{2x} - 0\right| < e^{-\frac{1}{2}x^2} < \epsilon$$

   - In case $0 < \epsilon \le 1$, we quickly solve the equation

$$e^{-\frac{1}{2}x^2} = \epsilon$$

     for x, yielding $x = \sqrt{-2\ln\epsilon}$. Choose $\tau = \max\left\{4, \sqrt{-2\ln\epsilon}\right\}$ then for $x > \tau$

$$\left|e^{-x^2}e^{2x} - 0\right| < e^{-\frac{1}{2}x^2} < e^{-\frac{1}{2}\tau^2} \le e^{-\frac{1}{2}\left(\sqrt{-2\ln\epsilon}\right)^2} = e^{\ln\epsilon} = \epsilon$$

22. Find

   c. $\lim_{x\to x_0} \frac{1}{(x-x_0)^{2k}}$, $\quad k$ is a positive integer.

   Solution

   Observe that in the extended reals

$$\lim_{x\to x_0-} \frac{1}{(x-x_0)^{2k}} = \infty \quad \text{and} \quad \lim_{x\to x_0+} \frac{1}{(x-x_0)^{2k}} = \infty$$

so

$$\lim_{x \to x_0} \frac{1}{(x - x_0)^{2k}} = \infty$$

d. $\lim_{x \to x_0} \frac{1}{(x-x_0)^{2k+1}}$, k is a positive integer.

Solution

Observe that in the extended reals

$$\lim_{x \to x_0 -} \frac{1}{(x - x_0)^{2k+1}} = -\infty \quad \text{and} \quad \lim_{x \to x_0 +} \frac{1}{(x - x_0)^{2k+1}} = \infty$$

so, even in the extended reals, the undirected limit

$$\lim_{x \to x_0} \frac{1}{(x - x_0)^{2k+1}}$$

does not exist.

2.2 Continuity

2. Prove that a function f is continuous at x_0 if and only if

$$\lim_{x \to x_0 -} f(x) = \lim_{x \to x_0 +} f(x) = f(x_0)$$

Solution

- If:

  Let $\epsilon > 0$. Since $\lim_{x \to x_0 -} f(x) = f(x_0)$ there exists a $\delta_1 > 0$ such that

  $$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 - \delta_1 < x \leq x_0$$

  Similarly, since $\lim_{x \to x_0 +} f(x) = f(x_0)$, there exists a $\delta_2 > 0$ such that

  $$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 \leq x < x_0 + \delta_2$$

  Choose $\delta = \min\{\delta_1, \delta_2\}$, then

  $$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

  This shows that f is continuous at x_0.

- Only if:
  Let $\epsilon > 0$. Since f is continuous at x_0, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

  This implies that

$$\begin{aligned} |f(x) - f(x_0)| &< \epsilon \quad \text{whenever} \quad x_0 - \delta < x \leq x_0, \text{ and} \\ |f(x) - f(x_0)| &< \epsilon \quad \text{whenever} \quad x_0 \leq x < x_0 + \delta \end{aligned}$$

  Hence $\lim_{x \to x_0-} f(x) = f(x_0)$ and $\lim_{x \to x_0+} f(x) = f(x_0)$.

3. Determine whether f is continuous from the left or from the right at x_0.

c. $f(x) = \frac{1}{x} \quad (x_0 = 0)$

Solution

Since $f(0)$ is undefined, the function f is neither continuous from the left at 0, nor continuous from the right at 0.

g. $f(x) = \begin{cases} \frac{x+|x|(1+x)}{x} \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad (x_0 = 0)$

Solution

Observe

$$\begin{aligned} \lim_{x \to 0-} f(x) &= \lim_{x \to 0-} \frac{x + |x|(1+x)}{x} \sin \frac{1}{x} = \lim_{x \to 0-} \frac{x - x(1+x)}{x} \sin \frac{1}{x} \\ &= \lim_{x \to 0-} \left(-x \sin \frac{1}{x} \right) = 0 \neq 1 = f(0) \end{aligned}$$

Hence, the function f is not continuous from the left at 0.

Note: The answer in the back of the book is not correct.

Similarly

$$\begin{aligned} \lim_{x \to 0+} f(x) &= \lim_{x \to 0+} \frac{x + |x|(1+x)}{x} \sin \frac{1}{x} \\ &= \lim_{x \to 0+} \frac{x + x(1+x)}{x} \sin \frac{1}{x} = \lim_{x \to 0+} (2+x) \sin \frac{1}{x} \end{aligned}$$

This limit is undefined. So, f is not continuous from the right at 0 either.

5. Let

$$g(x) = \frac{\sqrt{x}}{x-1}$$

On which of the following intervals is f continuous according to definition 2.2.3:

$[0,1), (0,1), (0,1], [1,\infty), (1,\infty)$?

Solution

$[0,1), (0,1)$, and $(1,\infty)$.

11. Prove that the function $g(x) = \log x$ is continuous on $(0,\infty)$. Take the following properties as given.

(a) $\lim_{x\to 1} g(x) = 0$

(b) $g(x_1) + g(x_2) = g(x_1 x_2)$ if $x_1, x_2 > 0$.

Solution

Let $\epsilon > 0$ and $x_0 \in (0,\infty)$. In class we showed that property (b) is equivalent to

$$g(x_1) - g(x_2) = g\left(\frac{x_1}{x_2}\right) \quad \text{if} \quad x_1, x_2 > 0$$

Let $x \in (0,\infty)$. Consider

$$|g(x) - g(x_0)| = \left|g\left(\frac{x}{x_0}\right)\right|$$

Since $\lim_{x\to 1} g(x) = 0$, there exists a $\delta_1 > 0$, such that

$$|g(u)| = |g(u) - 0| = |g(u) - g(1)| < \epsilon \quad \text{whenever} \quad |u-1| < \delta_1$$

Observe that

$$\left|\frac{x}{x_0} - 1\right| = \left|\frac{x - x_0}{x_0}\right| < \delta_1 \quad \text{whenever} \quad |x - x_0| < \delta_1 |x_0|$$

Choose $\delta = \delta_1 |x_0|$ then, by letting $\frac{x}{x_0}$ play the role of u, we may conclude that

$$|g(x) - g(x_0)| = \left|g\left(\frac{x}{x_0}\right)\right| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

Therefore g is continuous at x_0, and since x_0 was chosen arbitrarily on $(0,\infty)$ this shows that g is continuous on $(0,\infty)$.

16. Let $|f|$ be the function whose value at each x in D_f is $|f(x)|$. Prove: If f is continuous at x_0, then so is $|f|$. Is the converse true?

Solution

Let $\epsilon > 0$, then

$$||f|(x) - |f|(x_0)| = ||f(x)| - |f(x_0)|| \le |f(x) - f(x_0)|$$

Moreover, since f is continuous at x_0, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$, so

$$||f|(x) - |f|(x_0)| \le |f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

Hence, the function $|f|$ is continuous at x_0.

The converse is not true. Consider for instance the function

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then $|f|$ is continuous at 0, but f is not.

20. (a) Let f_1 and f_2 be continuous at x_0 and define

$$F(x) = \max\{f_1(x), f_2(x)\}$$

Show that F is continuous at x_0.

Solution

The key idea for this proof is to make a distinction between the case that $f_1(x_0) = f_2(x_0)$, and the case that $f_1(x_0) \neq f_2(x_0)$. Let $\epsilon > 0$.

- Case 1: $f_1(x_0) = f_2(x_0)$
  Note that in this case $F(x_0) = \max\{f_1(x_0), f_2(x_0)\} = f_1(x_0) = f_2(x_0)$. Since f_1 is continuous at x_0, there exists a δ_1 such that

  $$|f_1(x) - F(x_0)| = |f_1(x) - f_1(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta_1$$

  Similarly, since f_2 is continuous at x_0, there exists a δ_2 such that

  $$|f_2(x) - F(x_0)| = |f_2(x) - f_2(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta_2$$

  Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, because $F(x)$ either equals either $f_1(x)$ or $f_2(x)$,

  $$|F(x) - F(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

  This shows that F is continuous at x_0.

- Case 2: $f_1(x_0) \neq f_2(x_0)$
  Since f_1 is continuous at x_0, there exists a δ_1 such that

  $$|f_1(x) - f_1(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta_1$$

  Similarly, since f_2 is continuous at x_0, there exists a δ_2 such that

  $$|f_2(x) - f_2(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta_2$$

  Choose $\delta = \min\{\delta_1, \delta_2\}$. Without loss of generality, we may assume that $f_1(x_0) > f_2(x_0)$. Take $\epsilon < \frac{f_1(x_0) - f_2(x_0)}{2}$. This assures that whenever $|x - x_0| < \delta$

  $$|F(x) - F(x_0)| = |f_1(x) - f_1(x_0)| < \epsilon$$

  Again, this shows that F is continuous at x_0.

(b) Let $f_1, f_2, \ldots, f_n$ be continuous at x_0 and define

$$F(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

Show that F is continuous at x_0.

Solution

Use mathematical induction. Let P_n denote the proposition mentioned above. Part (a) of this exercise shows that P_2 is true. Let n denote any positive integer greater than or equal to 2 and assume that $P_2, P_3, \ldots P_n$ are all true. Let

$$h(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$$

then h and f_{n+1} are both continuous at x_0 and therefore

$$F(x) = \max\{f_1(x), f_2(x), \ldots, f_{n+1}(x)\} = \max\{h(x), f_{n+1}(x)\}$$

is continuous at x_0. Hence P_{n+1} is true and by the principle of mathematical induction we may conclude that P_n is true for all positive integers greater than or equal to 2.

21. Find the domains of $f \circ g$ and $g \circ f$.

a. $f(x) = \sqrt{x}$, $g(x) = 1 - x^2$

Solution

$D_f = [0, \infty)$ and $D_g = (-\infty, \infty)$.

- Let $T = [-1, 1]$. Then $T \subset D_g$ and $g(x) \in D_f$ whenever $x \in T$. The set T is the domain of $f \circ g$.

- Let $T = [0, \infty)$. Then $T \subset D_f$ and $f(x) \in D_g$ whenever $x \in T$. The set T is the domain of $g \circ f$.

c. $f(x) = \frac{1}{1-x^2}$, $g(x) = \cos x$

Solution

$D_f = \{x \mid x \neq -1, 1\}$ and $D_g = (-\infty, \infty)$.

- Let $T = \{x \mid x \neq n\pi, n \in \mathbb{Z}\}$. Then $T \subset D_g$ and $g(x) \in D_f$ whenever $x \in T$. The set T is the domain of $f \circ g$.
- Let $T = \{x \mid x \neq -1, 1\}$. Then $T \subset D_f$ and $f(x) \in D_g$ whenever $x \in T$. The set T is the domain of $g \circ f$.

23. Use Theorem 2.2.7 to find all points x_0 at which the following functions are continuous.

a. $\sqrt{1-x^2}$

Solution

Let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Then g is continuous at all real x_0, while f is continuous at all $x_0 > 0$. We conclude that $f \circ g$ is continuous at all x_0 with $1 - x_0^2 > 0$; that is the set

$$\left\{x_0 \mid 1 - x_0^2 > 0\right\} = (-1, 1)$$

g. $\left(1 - \sin^2 x\right)^{-1/2}$

Solution

Let $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 1 - \sin^2 x = \cos^2 x$. Then g is continuous at all real x_0, while f is continuous at all $x_0 > 0$. We conclude that $f \circ g$ is continuous at all x_0 with $\cos^2 x_0 > 0$; that is the set

$$\left\{x_0 \mid \cos^2 x_0 > 0\right\} = \left\{x_0 \mid \cos^2 x_0 \neq 0\right\} = \left\{x_0 \mid x_0 \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$$

Note: The answer in the back of the book is not correct.

24. Complete the proof of Theorem 2.2.9 by showing that there is an $x_2 \in [a, b]$ such that $f(x_2) = \beta$.

Solution

Recall that f is continuous on $[a, b]$ and $\beta = \sup_{x \in [a,b]} f(x) = \sup\{f(x) \mid x \in [a, b]\}$. Suppose there is no $x_2 \in [a, b]$ such that $f(x_2) = \beta$. Then $f(x) < \beta$ for all $x \in [a, b]$. Let $t \in [a, b]$, then $f(t) < \beta$, so

$$f(t) < \frac{f(t) + \beta}{2} < \beta$$

Moreover, since f is continuous at t, there exists an open interval I_t containing t, such that

$$f(x) < \frac{f(t)+\beta}{2} \text{ for all } x \in I_t \cap [a,b]$$

Note that $\mathcal{H} = \{I_t \mid t \in [a,b]\}$ is an open covering of $[a,b]$, and since $[a,b]$ is compact, $\mathcal{H}$ can be reduced to a finite sub-cover, say

$$\{I_{t_i} \mid 1 \le i \le n\}$$

Let

$$\beta_1 = \max_{1 \le i \le n} \frac{f(t_i)+\beta}{2}$$

Observe that $\beta_1 < \beta$, and

$$f(x) < \beta_1 \quad \text{for all} \quad x \in [a,b]$$

This implies that β_1 is an upper bound for the set $V = \{f(x) \mid x \in [a,b]\}$ which is less than $\beta = \sup V$, a contradiction. We conclude that there must be an $x_2 \in [a,b]$ such that $f(x_2) = \beta$.

2.3 Differentiable Functions of One Variable

3. Use Lemma 2.3.2. to prove that if $f'(x_0) > 0$, there is a $\delta > 0$ such that

$$f(x) < f(x_0) \text{ if } x_0 - \delta < x < x_0 \text{ and } f(x) > f(x_0) \text{ if } x_0 < x < x_0 + \delta$$

Solution

Since $\lim_{x \to x_0} E(x) = 0$, there exists a $\delta > 0$ such that $|E(x)| < f'(x_0)$ whenever $0 < |x - x_0| < \delta$. Therefore, if $0 < |x - x_0| < \delta$,

$$f'(x_0) + E(x) > 0$$

Moreover, since

$$f(x) - f(x_0) = \left[f'(x_0) + E(x)\right](x - x_0)$$

the desired result follows.

5. Find all derivatives of $f(x) = x^{n-1}|x|$, where n is a positive integer.

Solution

Recall that

$$\frac{d}{dx}|x| = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$$

- Using the product rule we find that if $x \neq 0$
  - $f'(x) = (n-1)\,x^{n-2}\,|x| + x^{n-1}\frac{|x|}{x} = (n-1)\,x^{n-2}\,|x| + x^{n-2}\,|x| = nx^{n-2}\,|x|$, and similarly
  - $f''(x) = n\,(n-1)\,x^{n-3}\,|x|$

    ...
  - $f^{(k)}(x) = n\,(n-1)\ldots(n-k+1)\,x^{n-k-1}\,|x|$ for $1 \leq k \leq n-1$

    Observe that the result for $f^{(k)}(x)$ is based on the premise that $f^{(k-1)}(x)$ is of the form $ax^b\,|x|$ where a and b are positive integers. The $(n-1)^{st}$derivative of f is no longer of this form

    $$f^{(n-1)}(x) = n\,(n-1)\ldots 2\,|x| = n!\,|x|$$

    Hence, its derivative needs to be examined separately
  - $f^{(n)}(x) = \begin{cases} -n! & \text{if } x < 0 \\ n! & \text{if } x > 0 \end{cases}$, and
  - $f^{(k)}(x) = 0$ if $k > n$.
- Next we examine $f^{(k)}(0)$. Observe that for integers m

$$\left(\frac{d}{dx}x^m\,|x|\right)\Bigg|_{x=0} = \begin{cases} \lim_{x\to 0}\frac{x^m|x|-0}{x-0} = \lim_{x\to 0} x^{(m-1)}\,|x| = 0 & \text{if } m \geq 1 \\ \text{undefined} & \text{if } m < 1 \end{cases}$$

- Combining the results for $x = 0$ and $x \neq 0$ we obtain
  - $f^{(k)}(x) = n\,(n-1)\ldots(n-k+1)\,x^{n-k-1}\,|x|$ for $1 \leq k \leq n-1$

    Note: The answer in the back of the book is not correct.
  - $f^{(n)}(x) = \begin{cases} -n! & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ n! & \text{if } x > 0 \end{cases}$
  - $f^{(k)}(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}$ for $k \geq n+1$.

10. Prove Theorem 2.3.4 **(b)**.

    If f and g are differentiable at x_0, then so is $f - g$ and

    $$(f-g)'(x_0) = f'(x_0) - g'(x_0)$$

    Solution

Observe

$$\begin{aligned}
\lim_{x\to x_0}\frac{(f-g)(x)-(f-g)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)-g(x)-[f(x_0)-g(x_0)]}{x-x_0}\\
&= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)-[g(x)-g(x_0)]}{x-x_0}\right]\\
&= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}-\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\\
&= f'(x_0)-g'(x_0)
\end{aligned}$$

We conclude that $(f-g)'(x_0)$ exists and equals $f'(x_0)-g'(x_0)$.

11. Prove Theorem 2.3.4 **(d)**.

If f and g are differentiable at x_0 and $g(x_0)\neq 0$, then f/g is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0)=\frac{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}{[g(x_0)]^2}$$

Solution

Observe

$$\begin{aligned}
\lim_{x\to x_0}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{\frac{f(x)}{g(x)}-\frac{f(x_0)}{g(x_0)}}{x-x_0}=\lim_{x\to x_0}\frac{f(x)\,g(x_0)-f(x_0)\,g(x)}{(x-x_0)\,g(x)\,g(x_0)}\\
&= \lim_{x\to x_0}\frac{f(x)\,g(x_0)-f(x_0)\,g(x_0)+f(x_0)\,g(x_0)-f(x_0)\,g(x)}{(x-x_0)\,g(x)\,g(x_0)}\\
&= \lim_{x\to x_0}\frac{[f(x)-f(x_0)]\,g(x_0)-f(x_0)\,[g(x)-g(x_0)]}{(x-x_0)\,g(x)\,g(x_0)}\\
&= \lim_{x\to x_0}\frac{\frac{f(x)-f(x_0)}{x-x_0}g(x_0)-f(x_0)\,\frac{g(x)-g(x_0)}{x-x_0}}{g(x)\,g(x_0)}\\
&= \frac{\left[\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right]g(x_0)-f(x_0)\left[\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\right]}{[\lim_{x\to x_0}g(x)]\,g(x_0)}
\end{aligned}$$

Since g is differentiable at x_0, g is continuous at x_0, so $\lim_{x\to x_0}g(x)=g(x_0)$, therefore

$$\lim_{x\to x_0}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(x_0)}{x-x_0}=\frac{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}{[g(x_0)]^2}$$

which proves the stated result.

15. a. Show that $f'_+(a) = f'(a+)$ if both quantities exist.

Solution

In class I suggested you use the Mean Value Theorem. We will do just that. If $f'(a+)$ exists, then f' exists on some open interval $(a, a+\delta)$. If the right hand derivative $f'_+(a)$ exists, then f is continuous from the right at a. Let $x^* \in (a, a+\delta)$, then f is continuous on $[a, x^*]$ and differentiable on (a, x^*), so the Mean Value Theorem applies and there exists a $c \in (a, x^*)$ such that

$$\frac{f(x^*) - f(a)}{x^* - a} = f'(c)$$

This implies

$$f'_+(a) = \lim_{x^* \to a+} \frac{f(x^*) - f(a)}{x^* - a} = \lim_{x^* \to a+} f'(c)$$

Finally, note that because $c \in (a, x^*)$, $c \to a+$ if $x^* \to a+$ and since $\lim_{x \to a+} f'(x)$ exists

$$\lim_{x^* \to a+} f'(c) = \lim_{x \to a+} f'(x) = f'(a+)$$

We conclude that $f'_+(a) = f'(a+)$.

b. Example 2.3.4 shows that $f'_+(a)$ may exist even if $f'(a+)$ does not. Give an example where $f'(a+)$ exists but $f'_+(a)$ does not.

Solution

Take a simple function which is not continuous from the right at a. For instance

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

then

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x-1}{x} \quad \text{is undefined}$$

while $f'(0+) = 1$.

c. Complete the following statement so it becomes a theorem, and prove the theorem: "If $f'(a+)$ exists and f is ____ at a, then $f'_+(a) = f'(a+)$."

Solution

"If $f'(a+)$ exists and f is continuous from the right at a, then $f'_+(a) = f'(a+)$."

Proof

If $f'(a+)$ exists, then f' exists on some open interval $(a, a+\delta)$. Let $x^* \in (a, a+\delta)$, then since f is continuous from the right at a, f is continuous on $[a, x^*]$ and differentiable on (a, x^*), so the Mean Value Theorem applies and there exists a $c \in (a, x^*)$ such that

$$\frac{f(x^*) - f(a)}{x^* - a} = f'(c)$$

This implies

$$\lim_{x^* \to a+} \frac{f(x^*) - f(a)}{x^* - a} = \lim_{x^* \to a+} f'(c)$$

Note that because $c \in (a, x^*)$, $c \to a+$ if $x^* \to a+$ and since $\lim_{x \to a+} f'(x)$ exists

$$\lim_{x^* \to a+} f'(c) = \lim_{x \to a+} f'(x) = f'(a+)$$

We conclude that $f'_+(a)$ exists and equals $f'(a+)$.

20. Let n be a positive integer and

$$f(x) = \frac{\sin nx}{n \sin x}, \quad x \neq k\pi \quad (k \text{ is integer}).$$

a. Define $f(k\pi)$ such that f is continuous at $k\pi$.

Solution

Let

$$\begin{aligned} f(k\pi) &= \lim_{x \to k\pi} f(x) = \lim_{x \to k\pi} \frac{\sin nx}{n \sin x} = \lim_{x \to k\pi} \frac{n \cos nx}{n \cos x} \\ &= \lim_{x \to k\pi} \frac{\cos nx}{\cos x} = \frac{\cos nk\pi}{\cos k\pi} = \frac{(-1)^{nk}}{(-1)^k} = (-1)^{(n-1)k} \end{aligned}$$

and redefine f as

$$f(x) = \begin{cases} \frac{\sin nx}{n \sin x} & x \neq k\pi, k \in \mathbb{Z} \\ (-1)^{(n-1)k} & x = k\pi, k \in \mathbb{Z} \end{cases}$$

b. Show that if $\overline{x}$ is a local extreme point of f, then

$$|f(\overline{x})| = \left[1 + \left(n^2 - 1\right) \sin^2 \overline{x}\right]^{-1/2}$$

HINT: *Express* $\sin nx$ *and* $\cos nx$ *in terms of* $f(x)$ *and* $f'(x)$, *and add their squares to obtain a useful identity.*

Solution

First consider the case that $\overline{x} \neq k\pi, k \in \mathbb{Z}$, then $f'(\overline{x}) = 0$. Observe that

$$\sin n\overline{x} = n f(\overline{x}) \sin \overline{x}$$

and

$$\begin{aligned} 0 &= f'(\overline{x}) = \frac{n \cos n\overline{x}}{n \sin \overline{x}} - \frac{\sin n\overline{x} \cos \overline{x}}{n \sin^2 \overline{x}} \\ &= \frac{\cos n\overline{x}}{\sin \overline{x}} - f(\overline{x}) \frac{\cos \overline{x}}{\sin \overline{x}} \end{aligned}$$

hence

$$\cos n\overline{x} = f(\overline{x})\cos\overline{x}$$

therefore

$$\begin{aligned} 1 &= \cos^2 n\overline{x} + \sin^2 n\overline{x} = [f(\overline{x})\cos\overline{x}]^2 + [nf(\overline{x})\sin\overline{x}]^2 \\ &= f^2(\overline{x})\left[\cos^2\overline{x} + n^2\sin^2\overline{x}\right] = f^2(\overline{x})\left[1 + \left(n^2 - 1\right)\sin^2\overline{x}\right] \end{aligned}$$

which implies that

$$|f(\overline{x})| = \left[1 + \left(n^2 - 1\right)\sin^2\overline{x}\right]^{-1/2}$$

Moreover, if $k \in \mathbb{Z}$, $|f(k\pi)| = \left|(-1)^{(n-1)k}\right| = 1$, so the formula

$$|f(\overline{x})| = \left[1 + \left(n^2 - 1\right)\sin^2\overline{x}\right]^{-1/2}$$

is true even if $\overline{x} = k\pi, k \in \mathbb{Z}$.

c. Show that $|f(x)| \leq 1$ for all x. For what values of x is equality attained?

Solution

For integer k, let I denote the closed and bounded interval $[(k-1)\pi, k\pi]$. Then f is continuous on I and by the Extreme Value Theorem f must have a minimum m and a maximum M on I. Therefore there exists a local extreme point $\overline{x}$ on I such that $m = f(\overline{x})$. Hence, by the result of Part **(b)** $|m| = |f(\overline{x})| \leq 1$. In a similar fashion we can prove that $|M| \leq 1$. This means that for all x in I

$$-1 \leq -|m| \leq m \leq f(x) \leq M \leq |M| \leq 1$$

so

$$|f(x)| \leq 1 \text{ for all } x \text{ in } I.$$

Because k was an arbitrary integer this implies that $|f(x)| \leq 1$ for all real x. If $n = 1$ equality is attained for all $x \in \mathbb{R}$, and if $n > 1$ equality is attained for $x = k\pi, k \in \mathbb{Z}$.

2.4 L'Hospital's Rule

In Exercises 2 - 40, *find the indicated limits.*

6. $\lim_{x\to 0} \frac{\log(1+x)}{x}$

Solution

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1$$

7. $\lim_{x\to\infty} e^x \sin e^{-x^2}$

Solution

$$\begin{aligned}
\lim_{x\to\infty} e^x \sin e^{-x^2} &= \lim_{x\to\infty} \frac{\sin e^{-x^2}}{e^{-x}} = \lim_{x\to\infty} \frac{-2xe^{-x^2}\cos e^{-x^2}}{-e^{-x}} \\
&= \lim_{x\to\infty} \frac{2x}{e^{x^2-x}} = \lim_{x\to\infty} \frac{2}{(2x-1)\, e^{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2}{(2x-1)\, e^{x(x-1)}} = 0
\end{aligned}$$

20. $\lim_{x\to0} (1+x)^{1/x}$

Solution

$$\lim_{x\to0} (1+x)^{1/x} = e^{\lim_{x\to0} \frac{\log(1+x)}{x}} = e^{\lim_{x\to0} \frac{\frac{1}{1+x}}{1}} = e^1 = e$$

23. $\lim_{x\to0+} x^\alpha \log x$

Solution

Observe that if $\alpha \le 0$, $\lim_{x\to0+} x^\alpha \log x = -\infty$. When $\alpha > 0$, then

$$\lim_{x\to0+} x^\alpha \log x = \lim_{x\to0+} \frac{\log x}{x^{-\alpha}} = \lim_{x\to0+} \frac{\frac{1}{x}}{-\alpha x^{-\alpha-1}} = \lim_{x\to0+} -\frac{1}{\alpha} x^\alpha = 0$$

26. $\lim_{x\to1+} \left(\frac{x+1}{x-1}\right)^{\sqrt{x^2-1}}$

Solution

$$\begin{aligned}
\lim_{x\to1+} \left(\frac{x+1}{x-1}\right)^{\sqrt{x^2-1}} &= e^{\lim_{x\to1+}\left[\sqrt{x^2-1}\log\left(\frac{x+1}{x-1}\right)\right]} \\
&= e^{\lim_{x\to1+}\left[\sqrt{x+1}\sqrt{x-1}(\log(x+1)-\log(x-1))\right]} \\
&= e^{\lim_{x\to1+}\left[\sqrt{x+1}\sqrt{x-1}\log(x+1)-\sqrt{x+1}\sqrt{x-1}\log(x-1)\right]} \\
&= e^{\lim_{x\to1+}\left[-\sqrt{2}\sqrt{x-1}\log(x-1)\right]} = e^{\lim_{x\to1+}\left[-\sqrt{2}\frac{\log(x-1)}{(x-1)^{-1/2}}\right]} \\
&= e^{\lim_{x\to1+}\left[2\sqrt{2}\frac{\frac{1}{x-1}}{(x-1)^{-3/2}}\right]} = e^{\lim_{x\to1+}\left[2\sqrt{2}\sqrt{x-1}\right]} = 1
\end{aligned}$$

2.5 Taylor's Theorem

2. Suppose $f^{(n+1)}(x_0)$ exists, and let T_n be the n^{th} Taylor polynomial of f about x_0. Show that the function

$$E_n(x) = \begin{cases} \frac{f(x)-T_n(x)}{(x-x_0)^n} & \text{if } x \in D_f - \{x_0\} \\ 0 & \text{if } x = 0 \end{cases}$$

Is differentiable at x_0 and find $E_n'(x_0)$.

Solution

Observe

$$\lim_{x\to x_0}\frac{E_n(x)-E_n(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{E_n(x)-0}{x-x_0}=\lim_{x\to x_0}\frac{\frac{f(x)-T_n(x)}{(x-x_0)^n}}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-T_n(x)}{(x-x_0)^{n+1}}$$

Let $T_{n+1}(x)$ denote the $(n+1)^{st}$ Taylor polynomial of f about x_0. Then

$$\begin{aligned}\lim_{x\to x_0}\frac{E_n(x)-E_n(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)-T_n(x)}{(x-x_0)^{n+1}}\\ &= \lim_{x\to x_0}\frac{f(x)-T_{n+1}(x)}{(x-x_0)^{n+1}}+\lim_{x\to x_0}\frac{T_{n+1}(x)-T_n(x)}{(x-x_0)^{n+1}}\\ &= 0+\lim_{x\to x_0}\left[\frac{1}{(x-x_0)^{n+1}}\left\{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}\right\}\right]\\ &= \frac{f^{(n+1)}(x_0)}{(n+1)!}\end{aligned}$$

We conclude that

$$E_n'(x_0)=\frac{f^{(n+1)}(x_0)}{(n+1)!}$$

4. a. Prove: if $f''(x_0)$ exists, then

$$\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}=f''(x_0)$$

Solution

Since $f''(x_0)$ exists, the second Taylor polynomial T_2 of f about x_0 is defined. We will compare $f(x_0+h)$ to $T_2(x_0+h)$, and $f(x_0+h)$ to $T_2(x_0+h)$. Recall

$$\lim_{x\to x_0}\frac{f(x)-T_2(x)}{(x-x_0)^2}=0$$

If in this result x is replaced by x_0+h we obtain

$$\lim_{h\to 0}\frac{f(x_0+h)-T_2(x_0+h)}{h^2}=0$$

and similarly

$$\lim_{h\to 0}\frac{f(x_0-h)-T_2(x_0-h)}{h^2}=0$$

Now observe

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}\\
=\;&\lim_{h\to 0}\frac{f(x_0+h)-T_2(x_0+h)}{h^2}+\lim_{h\to 0}\frac{T_2(x_0+h)-2f(x_0)+T_2(x_0-h)}{h^2}\\
&+\lim_{h\to 0}\frac{f(x_0-h)-T_2(x_0-h)}{h^2}\\
=\;&\lim_{h\to 0}\frac{T_2(x_0+h)-2f(x_0)+T_2(x_0-h)}{h^2}\\
=\;&\lim_{h\to 0}\left[\frac{f(x_0)+f'(x_0)h+1/2f''(x_0)h^2}{h^2}-\frac{2f(x_0)}{h^2}\right.\\
&\left.+\frac{f(x_0)-f'(x_0)h+1/2f''(x_0)h^2}{h^2}\right]\\
=\;&\lim_{h\to 0}\frac{f''(x_0)h^2}{h^2}=\lim_{h\to 0}f''(x_0)=f''(x_0)
\end{aligned}$$

b. Prove or give a counter example: If the limit in Part a exists, then so does $f''(x_0)$ and they are equal.

Solution

Notice that the proof of Part a is based on the assumption that $f''(x_0)$ exists. Without that assumption T_2 is undefined and the proof falls apart. To generate a counterexample for the given statement we will look for a function f that quadratically approaches $f(x_0)$ as x approaches x_0 and for which $f''(x_0)$ is undefined. With $x_0=0$, the function $f(x)=x|x|$ satisfies those requirements. We now verify that this function truly is a counterexample.

- $f'(0)=\lim_{x\to 0}\frac{x|x|-0}{x}=\lim_{x\to 0}|x|=0$, and if $x\neq 0$, $f'(x)=|x|+\frac{x|x|}{x}=2|x|$. So

$$\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{2|x|}{x}\quad\text{is undefined}$$

  Therefore $f''(0)$ is undefined.
- Observe

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(h)-2f(0)+f(-h)}{h^2}\\
=\;&\lim_{h\to 0}\frac{h|h|-h|-h|}{h^2}=\lim_{h\to 0}\frac{0}{h^2}=\lim_{h\to 0}0=0
\end{aligned}$$

This means that with $x_0 = 0$, $\lim_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = 0$.

8. a. Let

$$h(x) = \sum_{r=0}^{n} \alpha_r (x - x_0)^r$$

be a polynomial of degree $\leq n$ such that

$$\lim_{x\to x_0} \frac{h(x)}{(x-x_0)^n} = 0$$

Show that $\alpha_r = 0$ for $0 \leq r \leq n$.

Solution

We use induction on n. Let P_n denote the proposition above.

- First we verify that P_0 is true. Note that if $n = 0$, them $h(x) = \alpha_0$ and $(x - x_0)^n = 1$, so

$$\lim_{x\to x_0} \alpha_0 = 0$$

  Hence, $\alpha_0 = 0$.

- Let k denote a nonnegative integer and let P_k be true. Additionally let

$$h(x) = \sum_{r=0}^{k+1} \alpha_r (x - x_0)^r \quad \text{and} \quad \lim_{x\to x_0} \frac{h(x)}{(x-x_0)^{k+1}} = 0$$

  Then

$$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} \left[\frac{h(x)}{(x-x_0)^{k+1}} (x-x_0)^{k+1} \right] = 0 \cdot 0 = 0$$

  and since h is a polynomial it is continuous everywhere, so

$$h(x_0) = \lim_{x\to x_0} h(x) = 0$$

  which means $\alpha_0 = 0$. Thus, with $m = r - 1$

$$h(x) = \sum_{r=1}^{k+1} \alpha_r (x - x_0)^r = \sum_{m=0}^{k} \alpha_{m+1} (x - x_0)^{m+1}$$

  and

$$\begin{aligned} 0 &= \lim_{x\to x_0} \frac{h(x)}{(x-x_0)^{k+1}} = \lim_{x\to x_0} \frac{\sum_{m=0}^{k} \alpha_{m+1} (x-x_0)^{m+1}}{(x-x_0)^{k+1}} \\ &= \lim_{x\to x_0} \frac{\sum_{m=0}^{k} \alpha_{m+1} (x-x_0)^{m}}{(x-x_0)^{k}} \end{aligned}$$

Observe that the numerator in the last expression is a polynomial of degree $\leq k$, so the induction assumption applies and we may conclude that in addition to the fact that $\alpha_0 = 0$, also $\alpha_r = 0$ for $1 \leq r \leq k+1$. Hence P_{k+1} is true. This completes the proof.

b. Suppose f is n times differentiable at x_0 and $p = \sum_{r=0}^{n} \alpha_r (x - x_0)^r$ is a polynomial of degree $\leq n$ such that

$$\lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^n} = 0$$

Show that

$$\alpha_r = \frac{f^{(r)}(x_0)}{r!} \quad \text{if} \quad 0 \leq r \leq n$$

that is, $p = T_n$. the n^{th} Taylor polynomial of f about x_0.

Solution

Observe

$$\begin{aligned} 0 &= \lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} + \lim_{x \to x_0} \frac{T_n(x) - p(x)}{(x - x_0)^n} \\ &= 0 + \lim_{x \to x_0} \frac{T_n(x) - p(x)}{(x - x_0)^n} \end{aligned}$$

we conclude that

$$\lim_{x \to x_0} \frac{T_n(x) - p(x)}{(x - x_0)^n} = 0$$

and by Part a, this implies that $T_n(x) - p(x)$ is identically equal to zero, so $p = T_n$.

16. Find an upper bound for the magnitude of the error in the approximation.

b. $\sqrt{1+x} \approx 1 + \frac{x}{2}$, $|x| < \frac{1}{8}$

Solution

Let $f(x) = \sqrt{1+x}$, then use Taylor's theorem with $n = 1$ and $x_0 = 0$.

$$\sqrt{1+x} - \left[1 + \frac{x}{2}\right] = \frac{f^{(2)}(c)}{2!} x^2$$

so

$$\left|\sqrt{1+x} - \left[1 + \frac{x}{2}\right]\right| = \left|\frac{f^{(2)}(c)}{2!} x^2\right| \leq \frac{1}{2}\left(\frac{1}{8}\right)^2 \left|f^{(2)}(c)\right| = \frac{1}{128}\left|f^{(2)}(c)\right|$$

Next we estimate $\left|f^{(2)}(c)\right| = \frac{1}{4(c+1)^{3/2}}$ for $|c| < \frac{1}{8}$. Observe

$$\left|f^{(2)}(c)\right| = \frac{1}{4(c+1)^{3/2}} \leq \frac{1}{4\left(-\frac{1}{8} + 1\right)^{3/2}} = \frac{4}{49}\sqrt{14}$$

Hence, an upper bound for the magnitude of the error is given by

$$\frac{1}{128} \cdot \frac{4}{49}\sqrt{14} = \frac{1}{1568}\sqrt{14} \approx 2.3863 \times 10^{-3}$$

Note: The answer in the back of the book is not correct.

d. $\log x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$, $|x-1| < \frac{1}{64}$

Solution

Let $f(x) = \log x$, then use Taylor's theorem with $n = 3$ and $x_0 = 1$.

$$\log x - \left[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right] = \frac{f^{(4)}(c)}{4!}(x-1)^4$$

so

$$\begin{aligned} &\left|\log x - \left[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}\right]\right| \\ = &\left|\frac{f^{(4)}(c)}{4!}(x-1)^4\right| \le \frac{1}{24}\left(\frac{1}{64}\right)^4 \left|f^{(4)}(c)\right| = \frac{1}{402\,653\,184}\left|f^{(4)}(c)\right| \end{aligned}$$

Next we estimate $\left|f^{(4)}(c)\right| = \frac{6}{c^4}$ for $|c-1| < \frac{1}{64}$. Observe

$$\left|f^{(4)}(c)\right| = \frac{6}{c^4} \le \frac{6}{\left(1-\frac{1}{64}\right)^4} = \frac{33\,554\,432}{5250\,987}$$

Hence, an upper bound for the magnitude of the error is given by

$$\frac{1}{402\,653\,184} \cdot \frac{33\,554\,432}{5250\,987} = \frac{1}{63\,011\,844} \approx 1.587 \times 10^{-8}$$