Chapter 3

Integral Calculus of Functions of One Variable

3.1 Definition of the Integral

1. Show that there cannot be more than one number L that satisfies Definition 3.1.1.

   Let f be defined on $[a, b]$. We say that f is Riemann integrable on $[a, b]$ if there is a number L with the following property: For every $\epsilon > 0$, there is a $\delta > 0$ such that

   $$|\sigma - L| < \epsilon$$

   if σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$.

   Solution

   Let $\epsilon > 0$. Suppose the numbers L_1 and L_2 both satisfy this definition. Then there exists a $\delta_1 > 0$ such that

   $$|\sigma - L_1| < \frac{\epsilon}{2}$$

   if σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta_1$. Similarly there exists a $\delta_2 > 0$ such that

   $$|\sigma - L_2| < \frac{\epsilon}{2}$$

   if σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$, then if σ is any Riemann sum of f over a partition P of $[a, b]$ such that $\|P\| < \delta$

   $$|L_2 - L_1| = |\sigma - L_1 - (\sigma - L_2)| \leq |\sigma - L_1| + |\sigma - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   Since this inequality holds for all $\epsilon > 0$, L_1 must equal L_2.

2. a. Prove: If $\int_a^b f(x)\,dx$ exists, then for every $\epsilon > 0$, there is a $\delta > 0$ such that $|\sigma_1 - \sigma_2| < \epsilon$ if σ_1 and σ_2 are Riemann sums of f over partitions P_1 and P_2 of $[a,b]$ with norms less than δ.

   Solution

   Let $L = \int_a^b f(x)\,dx$ and $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|\sigma - L| < \frac{\epsilon}{2}$$

   if σ is any Riemann sum of f over a partition P of $[a,b]$ such that $\|P\| < \delta$. This means that if σ_1 and σ_2 are Riemann sums of f over partitions P_1 and P_2 of $[a,b]$ with norms less than δ

$$|\sigma_1 - L| < \frac{\epsilon}{2} \text{ and } |\sigma_2 - L| < \frac{\epsilon}{2}$$

   hence

$$|\sigma_1 - \sigma_2| = |\sigma_1 - L - (\sigma_2 - L)| \le |\sigma_1 - L| + |\sigma_2 - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   b. Suppose that there is an $M > 0$ such that, for every $\delta > 0$, there are Riemann sums σ_1 and σ_2 over a partition P of $[a,b]$ with $\|P\| < \delta$ such that $|\sigma_1 - \sigma_2| \ge M$. Use Part (a) to prove that f is not integrable over $[a,b]$.

   Solution

   Suppose that f is integrable over $[a,b]$. Take $\epsilon < M$ and let δ be determined as in Part (a). Then with $P_1 = P_2 = P$ and $\|P\| < \delta$

$$|\sigma_1 - \sigma_2| < \epsilon < M$$

   for any two Riemann sums σ_1 and σ_2 of f over P, a contradiction.

4. Prove directly from Definition 3.1.1 that

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$$

Do not assume in advance that the integral exists. The proof of this is part of the problem. HINT: *Let* $P = \{x_0, x_{1,\dots,} x_n\}$ *be an arbitrary partition of* $[a,b]$. *Use the mean value theorem to show that*

$$\frac{b^3 - a^3}{3} = \sum_{j=1}^{n} d_j^2 (x_j - x_{j-1})$$

for some points $d_1, \dots, d_n$, *where* $x_{j-1} < d_j < x_j$. *Then relate this sum to arbitrary Riemann sums for* $f(x) = x^2$ *over* P.

Solution

Let $\epsilon > 0$. Apply the mean value theorem to the function $g(x) = x^3$ on the interval $[x_{j-1}, x_j]$. That is: there exists a point d_j with $x_{j-1} < d_j < x_j$ such that

$$\frac{g(x_j) - g(x_{j-1})}{x_j - x_{j-1}} = g'(d_j)$$

which implies

$$\frac{x_j^3 - x_{j-1}^3}{x_j - x_{j-1}} = 3d_j^2 \quad \text{so} \quad \frac{1}{3}\left(x_j^3 - x_{j-1}^3\right) = d_j^2 (x_j - x_{j-1})$$

Summing both sides of the last equation yields

$$\frac{b^3 - a^3}{3} = \sum_{j=1}^{n} \frac{1}{3}\left(x_j^3 - x_{j-1}^3\right) = \sum_{j=1}^{n} d_j^2 (x_j - x_{j-1})$$

An arbitrary Riemann sum of f over P is of the form

$$\begin{aligned}
\sigma &= \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} c_j^2 (x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} d_j^2 (x_j - x_{j-1}) + \sum_{j=1}^{n} \left(d_j^2 - c_j^2\right)(x_j - x_{j-1}) \\
&= \frac{b^3 - a^3}{3} + \sum_{j=1}^{n} \left(d_j^2 - c_j^2\right)(x_j - x_{j-1})
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\sigma - \frac{b^3 - a^3}{3}\right| &= \left|\sum_{j=1}^{n} \left(d_j^2 - c_j^2\right)(x_j - x_{j-1})\right| \le \sum_{j=1}^{n} \left|d_j^2 - c_j^2\right| (x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} |d_j - c_j| \, |d_j + c_j| \, (x_j - x_{j-1}) \le \sum_{j=1}^{n} |d_j - c_j| \left(|d_j| + |c_j|\right)(x_j - x_{j-1}) \\
&\le 2\|P\| \max\{|a|, |b|\} \sum_{j=1}^{n} (x_j - x_{j-1}) = 2\|P\| \max\{|a|, |b|\}(b - a)
\end{aligned}$$

Now choose $0 < \delta < \frac{\epsilon}{2(b-a)\max\{|a|,|b|\}}$, then

$$\left|\sigma - \frac{b^3 - a^3}{3}\right| < \epsilon$$

whenever $\|P\| < \delta$. This completes the proof.

7. Let f be bounded on $[a, b]$ and let P be a partition of $[a, b]$. Prove: The lower sum $s(P)$ of f over P is the infimum of the set of all Riemann sums of f over P.

Solution

If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then

$$s(P) = \sum_{j=1}^{n} m_j (x_j - x_{j-1})$$

where

$$m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

An arbitrary Riemann sum of f over P is of the form

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1})$$

where $x_{j-1} \le c_j \le x_j$. Since $m_j \le f(c_j)$, it follows that $s(P) \le \sigma$. Hence $s(P)$ is a lower bound for the set of all Riemann sums of f over P. To show that it is the greatest lower bound of this set, we let $\epsilon > 0$ and prove that there exists a Riemann sum $\overline{\sigma}$ of f over P such that $\overline{\sigma} < s(P) + \epsilon$. Because $m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$, the number $m_j + \frac{\epsilon}{n(x_j - x_{j-1})}$ is not a lower bound of f on $[x_{j-1}, x_j]$. Hence, there exists a $\overline{c_j} \in [x_{j-1}, x_j]$ with the property that

$$f(\overline{c_j}) < m_j + \frac{\epsilon}{n(x_j - x_{j-1})}$$

The Riemann sum $\overline{\sigma}$ produced this way is

$$\begin{aligned} \overline{\sigma} &= \sum_{j=1}^{n} f(\overline{c_j})(x_j - x_{j-1}) < \sum_{j=1}^{n} \left(m_j + \frac{\epsilon}{n(x_j - x_{j-1})} \right)(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} m_j (x_j - x_{j-1}) + \sum_{j=1}^{n} \frac{\epsilon}{n} = s(P) + \epsilon \end{aligned}$$

This completes the proof.

14. Suppose that $-\infty < a < d < b < \infty$ and

$$g(x) = \begin{cases} g_1, & a < x < d \\ g_2, & d < x < b \end{cases}$$

where g_1 and g_2 are constants, and let $g(a), g(b)$, and $g(d)$ be arbitrary. Suppose that f is defined on $[a, b]$, continuous from the right at a and form the left at b, and continuous at d. Show that $\int_a^b f(x)\,dg(x)$ exists and find its value.

Solution

Let $\epsilon > 0$ and let $P = \{x_0, x_1, \ldots, x_n\}$ denote a partition of $[a, b]$. In order to be able to separate the influence of the jump discontinuities of the function g on a Riemann-Stieltjes sum of f with respect to g over P, we let $\delta_1 = \frac{1}{2}\min\{d-a, b-d\}$ and $\|P\| \le \delta_1$. Then there exists an i with

$$2 \le i \le n-2$$

such that $d \in [x_i, x_{i+1})$. Hence an arbitrary Riemann-Stieltjes sum of f with respect to g over P takes the form

$$\begin{aligned}
\sigma &= \sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] = f(c_1)[g(x_1) - g(x_0)] + f(c_i)[g(x_i) - g(x_{i-1})] \\
&\quad + f(c_{i+1})[g(x_{i+1}) - g(x_i)] + f(c_n)[g(x_n) - g(x_{n-1})] \\
&= f(c_1)[g_1 - g(a)] + f(c_i)[g(x_i) - g_1] + f(c_{i+1})[g_2 - g(x_i)] + f(c_n)[g(b) - g_2] \\
&= f(a)[g_1 - g(a)] + f(d)[g(x_i) - g_1] + f(d)[g_2 - g(x_i)] + f(b)[g(b) - g_2] \\
&\quad + (f(c_1) - f(a))[g_1 - g(a)] + (f(c_i) - f(d))[g(x_i) - g_1] \\
&\quad + (f(c_{i+1}) - f(d))[g_2 - g(x_i)] + (f(c_n) - f(b))[g(b) - g_2]
\end{aligned}$$

where $c_j \in [x_{j-1}, x_j]$ for $1 \le j \le n$. Observe that if $d = x_i$, then

$$\begin{aligned}
f(d)[g(x_i) - g_1] + f(d)[g_2 - g(x_i)] &= f(d)[g(d) - g_1] + f(d)[g_2 - g(d)] \\
&= f(d)[g_2 - g_1]
\end{aligned}$$

and if $d \ne x_i$, then again

$$\begin{aligned}
f(d)[g(x_i) - g_1] + f(d)[g_2 - g(x_i)] &= f(d)[g_1 - g_1] + f(d)[g_2 - g_1] \\
&= f(d)[g_2 - g_1]
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma &= f(a)[g_1 - g(a)] + f(d)[g_2 - g_1] + f(b)[g(b) - g_2] \\
&\quad + (f(c_1) - f(a))[g_1 - g(a)] + (f(c_i) - f(d))[g(x_i) - g_1] \\
&\quad + (f(c_{i+1}) - f(d))[g_2 - g(x_i)] + (f(c_n) - f(b))[g(b) - g_2]
\end{aligned}$$

and because either $g(x_i) = g_1$, or $g(x_i) = g(d)$

$$\begin{aligned}
& |\sigma - \{f(a)[g_1 - g(a)] + f(d)[g_2 - g_1] + f(b)[g(b) - g_2]\}| \\
\leq\; & |f(c_1) - f(a)|\,|g_1 - g(a)| + |f(c_i) - f(d)|\,|g(x_i) - g_1| \\
& + |f(c_{i+1}) - f(d)|\,|g_2 - g(x_i)| + |f(c_n) - f(b)|\,|g(b) - g_2| \\
\leq\; & |f(c_1) - f(a)|\,(|g_1| + |g(a)|) + |f(c_i) - f(d)|\,(|g(d)| + |g_2|) \\
& + |f(c_{i+1}) - f(d)|\,(|g_2| + |g_1| + |g(d)|) + |f(c_n) - f(b)|\,(|g(b)| + |g_2|)
\end{aligned}$$

Just like we did in class, we use the continuity of f to estimate the expression on the right. Since $c_1 \geq a$ and $c_n \leq b$ and f is continuous from the right at a and from the left at b, and continuous at d, there exists a $\delta_2 > 0$ such that if

$$|c_1 - a|, |c_i - d|, |c_{i+1} - d|, \text{ and } |c_n - b|$$

are all less than δ_2, then

$$\begin{aligned}
& |f(c_1) - f(a)|\,(|g_1| + |g(a)|), |f(c_i) - f(d)|\,(|g(d)| + |g_2|), \\
& |f(c_{i+1}) - f(d)|\,(|g_2| + |g_1| + |g(d)|), \text{ and } |f(c_n) - f(b)|\,(|g(b)| + |g_2|)
\end{aligned}$$

are all less than $\frac{\epsilon}{4}$. Let $\delta = \min\left\{\delta_1, \frac{1}{2}\delta_2\right\}$, then

$$|\sigma - \{f(a)[g_1 - g(a)] + f(d)[g_2 - g_1] + f(b)[g(b) - g_2]\}| < \epsilon$$

whenever $\|P\| < \delta$. This shows that the Riemann-Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists and equals

$$\int_a^b f(x)\,dg(x) = f(a)[g_1 - g(a)] + f(d)[g_2 - g_1] + f(b)[g(b) - g_2]$$

3.2 Existence of the Integral

1. Complete the proof of Lemma 3.2.1 by verifying Eqn. (3). That is:

   Suppose that
   $$|f(x)| \leq M, \quad a \leq x \leq b,$$
   and let P' be a partition of $[a, b]$ obtained by adding r points to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. Then show that
   $$s(P) \leq s(P') \leq s(P) + 2Mr\,\|P\|$$

   Solution

The proof of this result is almost identical to the proof of the first part of Lemma 3.2.1, which was covered in class. First we suppose that $r = 1$, so P' is obtained from P by adding one partition point c. Let $x_{i-1} < c < x_i$, and

$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x), \; m_{i1} = \inf_{x_{i-1} \le x \le c} f(x), \text{ and } m_{i2} = \inf_{c \le x \le x_i} f(x)$$

then

$$\begin{aligned} s\left(P'\right) - s(P) &= m_{i1}(c - x_{i-1}) + m_{i2}(x_i - c) - m_i(x_i - x_{i-1}) \\ &= (m_{i1} - m_i)(c - x_{i-1}) + (m_{i2} - m_i)(x_i - c) \end{aligned}$$

Since $m_{i1} \ge m_i$ and $m_{i2} \ge m_i$, this implies that $s(P') - s(P) \ge 0$. Hence

$$\begin{aligned} 0 &\le s\left(P'\right) - s(P) = (m_{i1} - m_i)(c - x_{i-1}) + (m_{i2} - m_i)(x_i - c) \\ &\le 2M(c - x_{i-1}) + 2M(x_i - c) = 2M(x_i - x_{i-1}) \le 2M\|P\| \end{aligned}$$

This establishes the desired result for $r = 1$. Now suppose that $r > 1$. Let $P^{(0)} = P$ and let $P^{(j)}$ be obtained by adding the point c_j to $P^{(j-1)}$. Finally, let $P' = P^{(r)}$, then

$$0 \le s\left(P^{(j)}\right) - s\left(P^{(j-1)}\right) \le 2M\left\|P^{(j-1)}\right\|, \quad 1 \le j \le r$$

Therefore

$$0 \le \sum_{j=1}^{r}\left[s\left(P^{(j)}\right) - s\left(P^{(j-1)}\right)\right] \le 2M\sum_{j=1}^{r}\left\|P^{(j-1)}\right\| \le 2Mr\|P\|$$

We conclude that

$$0 \le s\left(P'\right) - s(P) \le 2Mr\|P\|$$

which completes the proof.

2. Show that if f is integrable on $[a, b]$, then

$$\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

Solution

This is the second part of Theorem 3.2.3. Its proof is very similar to the proof of the first part of the theorem, which was presented in class. We compare the lower integral to a lower sum, then we compare the lower sum to a Riemann sum, and finally we compare the Riemann sum to the Riemann integral. Let $\epsilon > 0$. Suppose that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ and that σ is a Riemann sum of f over P. Then

$$\begin{aligned}\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| &\leq \left|\underline{\int_a^b} f(x)\,dx - s(P)\right| + |s(P) - \sigma| \\ &\quad + \left|\sigma - \int_a^b f(x)\,dx\right|\end{aligned}$$

Since the lower integral $\underline{\int_a^b} f(x)\,dx$ is the supremum of the lower sums, there exists a partition P_0 such that

$$\underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{3} < s(P_0) \leq \underline{\int_a^b} f(x)\,dx$$

Also, because f is integrable on$[a, b]$, there exists a $\delta > 0$ such that

$$\left|\sigma - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{3}$$

whenever $\|P\| < \delta$. Suppose that $\|P\| < \delta$ and P is a refinement of P_0. Then

$$\underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{3} < s(P_0) \leq s(P) \leq \underline{\int_a^b} f(x)\,dx$$

so

$$\left|\underline{\int_a^b} f(x)\,dx - s(P)\right| < \frac{\epsilon}{3}$$

Hence

$$\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{2\epsilon}{3} + |s(P) - \sigma|$$

and this inequality holds for every Riemann sum σ of f over P. Finally, since $s(P)$ is the infimum of these Riemann sums, we may choose σ in such a way that

$$|s(P) - \sigma| < \frac{\epsilon}{3}$$

and we obtain

$$\left|\underline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \epsilon$$

Since this is true for all $\epsilon > 0$, the desired result follows.

4. Prove: If f is integrable on $[a,b]$ and $\epsilon > 0$, then $S(P) - s(P) < \epsilon$ if $\|P\|$ is sufficiently small. HINT: *Use Theorem 3.1.4.*

Solution

Actually, the hint given here is a misprint. Instead you should use Lemma 3.2.4.

If f is bounded on $[a,b]$ and $\epsilon > 0$, there is a $\delta > 0$ such that

$$\overline{\int_a^b} f(x)\,dx \le S(P) < \overline{\int_a^b} f(x)\,dx + \epsilon$$

and

$$\underline{\int_a^b} f(x)\,dx \ge s(P) > \underline{\int_a^b} f(x)\,dx - \epsilon$$

if $\|P\| < \delta$.

Let $\epsilon > 0$. Since f is integrable on $[a,b]$, f is bounded on $[a,b]$, and

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

We now apply Lemma 3.2.4 with ϵ replaced by $\frac{\epsilon}{2}$. Then we know that there is a $\delta > 0$ such that for all partitions P of $[a,b]$ with $\|P\| < \delta$

$$\begin{aligned} S(P) - s(P) &< \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} - \left(\underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2}\right) \\ &= \int_a^b f(x)\,dx + \frac{\epsilon}{2} - \left(\int_a^b f(x)\,dx - \frac{\epsilon}{2}\right) = \epsilon \end{aligned}$$

This completes the proof.

3.3 Properties of the Integral

1. Prove Theorem 3.3.2

If f is integrable on $[a,b]$ and c is a constant, then cf is integrable on $[a,b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Solution

Let $\epsilon > 0$. Any Riemann sum of cf over a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ can be written as

$$\sigma_{cf} = \sum_{j=1}^{n} cf\left(c_j\right)\left(x_j - x_{j-1}\right) = c\sum_{j=1}^{n} f\left(c_j\right)\left(x_j - x_{j-1}\right) = c\sigma_f$$

If $c \neq 0$, then because f is integrable on $[a, b]$, there exists a $\delta > 0$ such that whenever $\|P\| < \delta$

$$\left|\sigma_f - \int_a^b f\left(x\right)dx\right| < \frac{\epsilon}{|c|}$$

so

$$\left|\sigma_{cf} - c\int_a^b f\left(x\right)dx\right| = \left|c\sigma_f - c\int_a^b f\left(x\right)dx\right| = |c|\left|\sigma_f - \int_a^b f\left(x\right)dx\right| < |c|\frac{\epsilon}{|c|} = \epsilon$$

This proves the desired result when $c \neq 0$. If $c = 0$, then both σ_{cf} and $c\int_a^b f\left(x\right)dx$ equal zero, so again

$$\left|\sigma_{cf} - c\int_a^b f\left(x\right)dx\right| = 0 < \epsilon$$