Platonic Solids,

or, the power of counting

Keith Jones

Pi Mu Epsilon Induction
SUNY Oneonta April 2017

What is a Platonic Solid?

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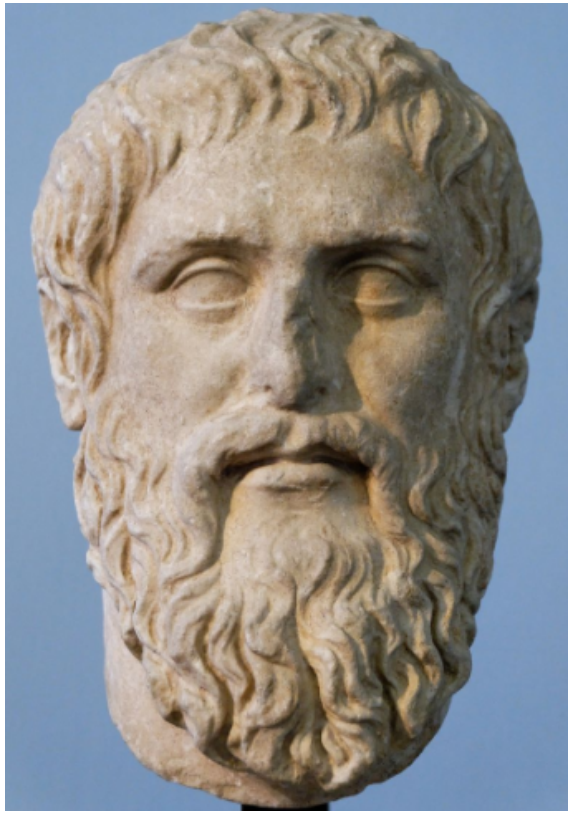

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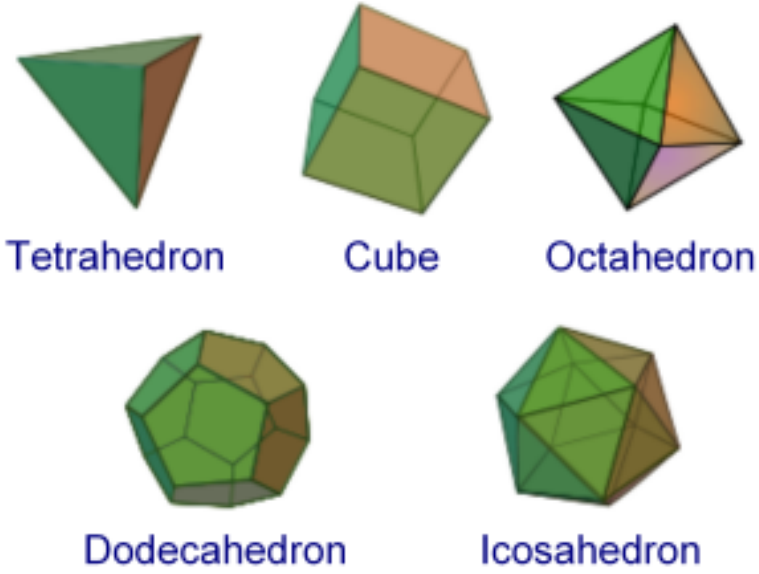
Named after, Plato, Platonic solids are polyhedrons (3-dimensional objects with flat faces and straight edges) satisfying:



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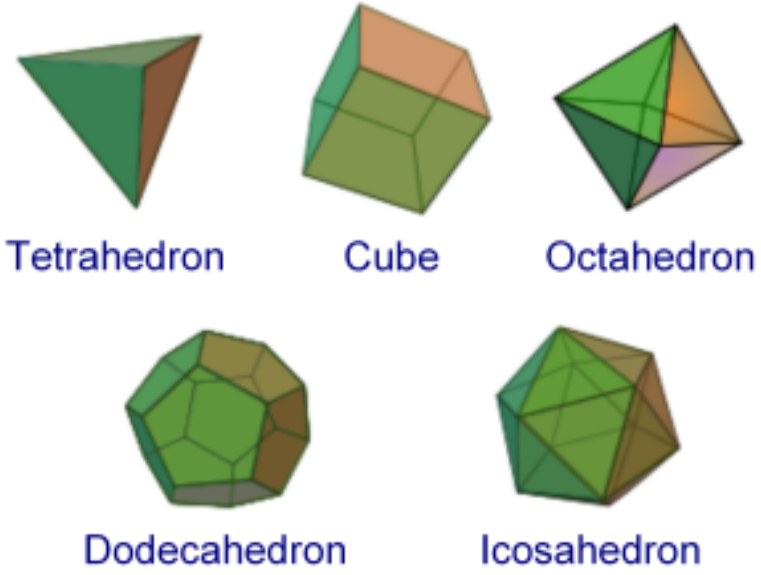

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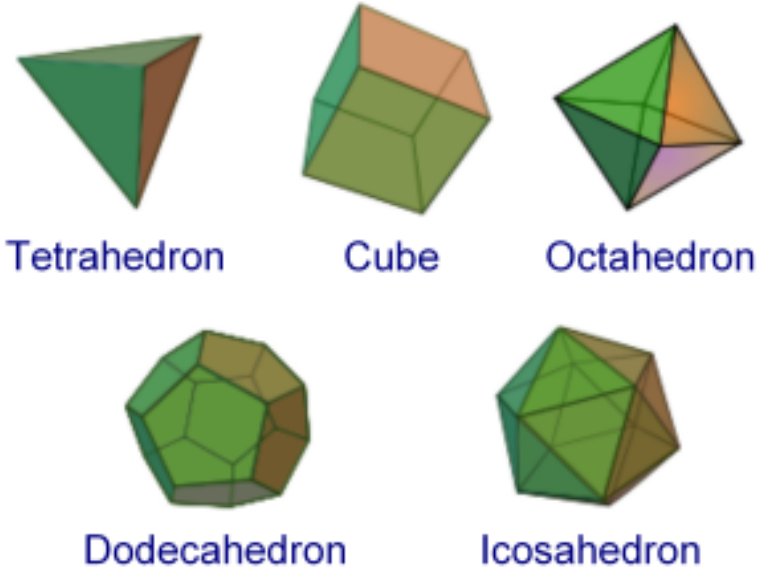

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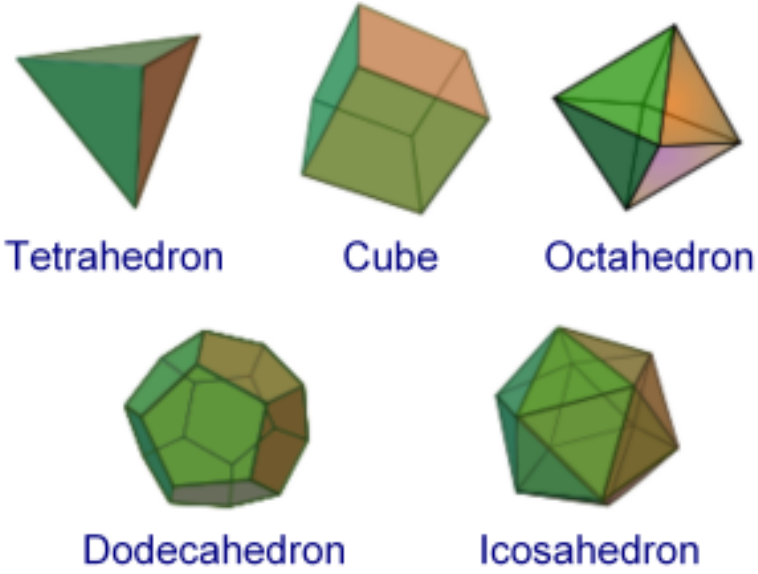

- all faces are congruent – same size and shape,
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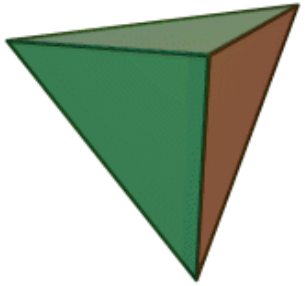
- the same number of faces meet at each vertex, and
- the solid is convex (no indentations).

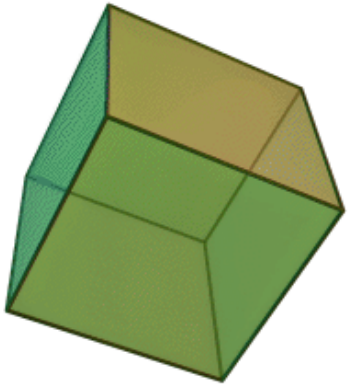
The Platonic Solids

1. The tetrahedron has 4 triangular faces, 6 edges, and 4 vertices:



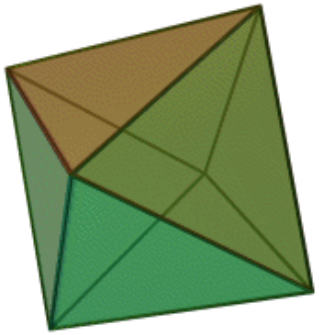
The Platonic Solids

2. The cube or hexahedron has 6 square faces, 12 edges, and 8 vertices:



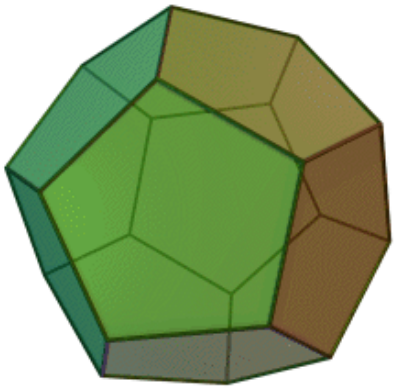
The Platonic Solids

3. The octahedron has 8 triangular faces, 12 edges, and 6 vertices:



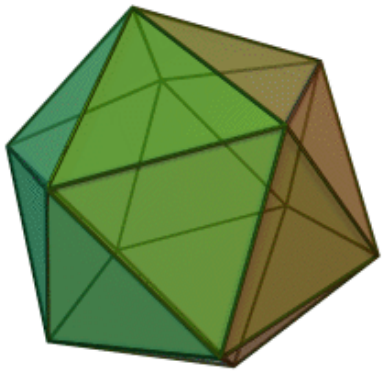
The Platonic Solids

4. The dodecahedron has 12 pentagonal faces, 30 edges, and 20 vertices:



The Platonic Solids

5. The icosahedron has 20 triangular faces, 30 edges, and 12 vertices:



A Bit of History...

Plato linked the solids to the four elements in his text *Timaeus* (360 BC), which speculates on the nature of the physical world.

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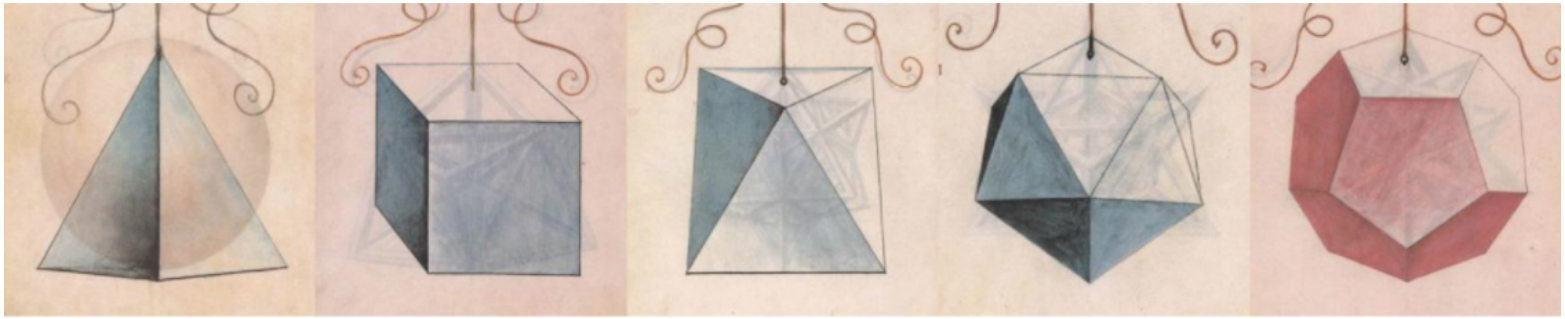

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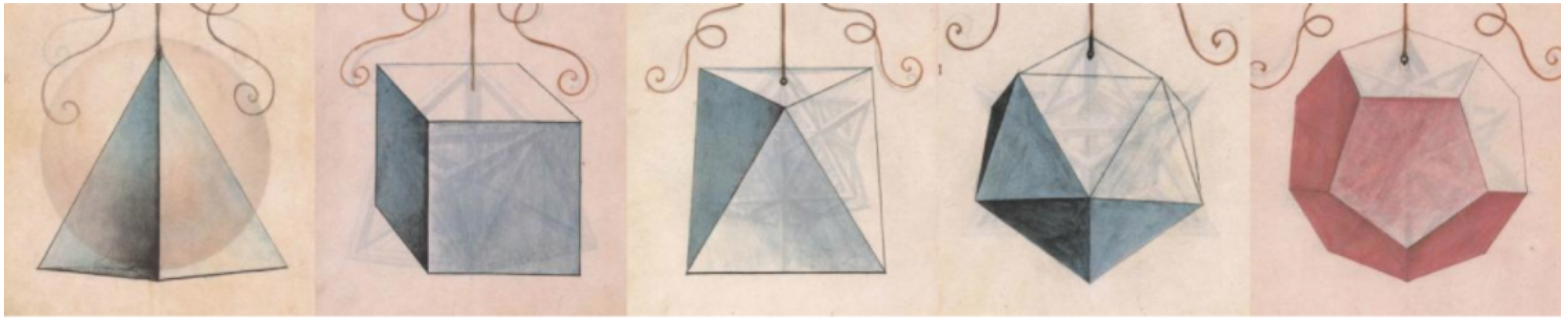

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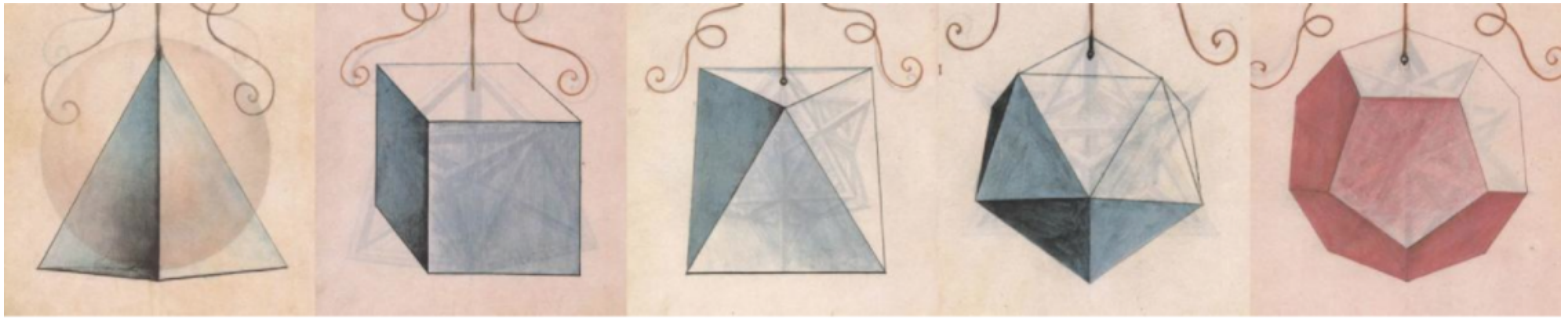

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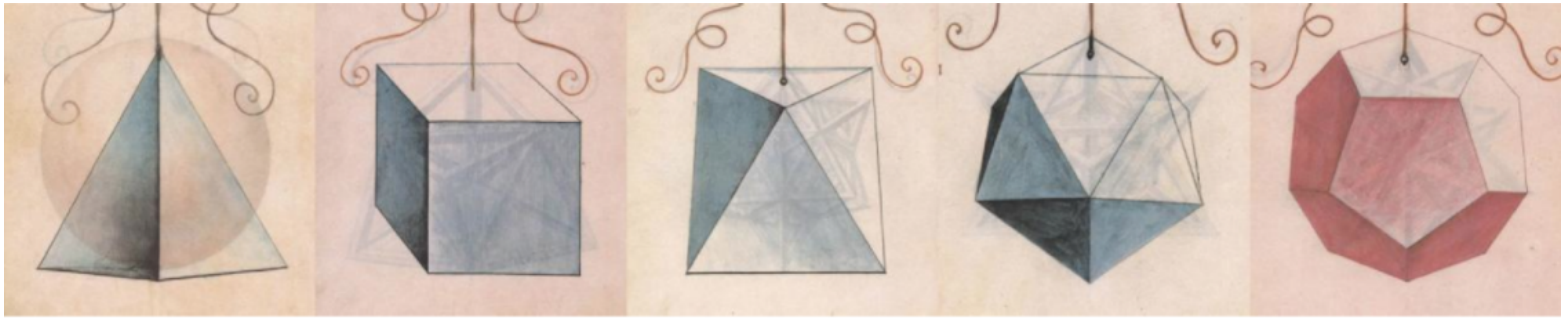
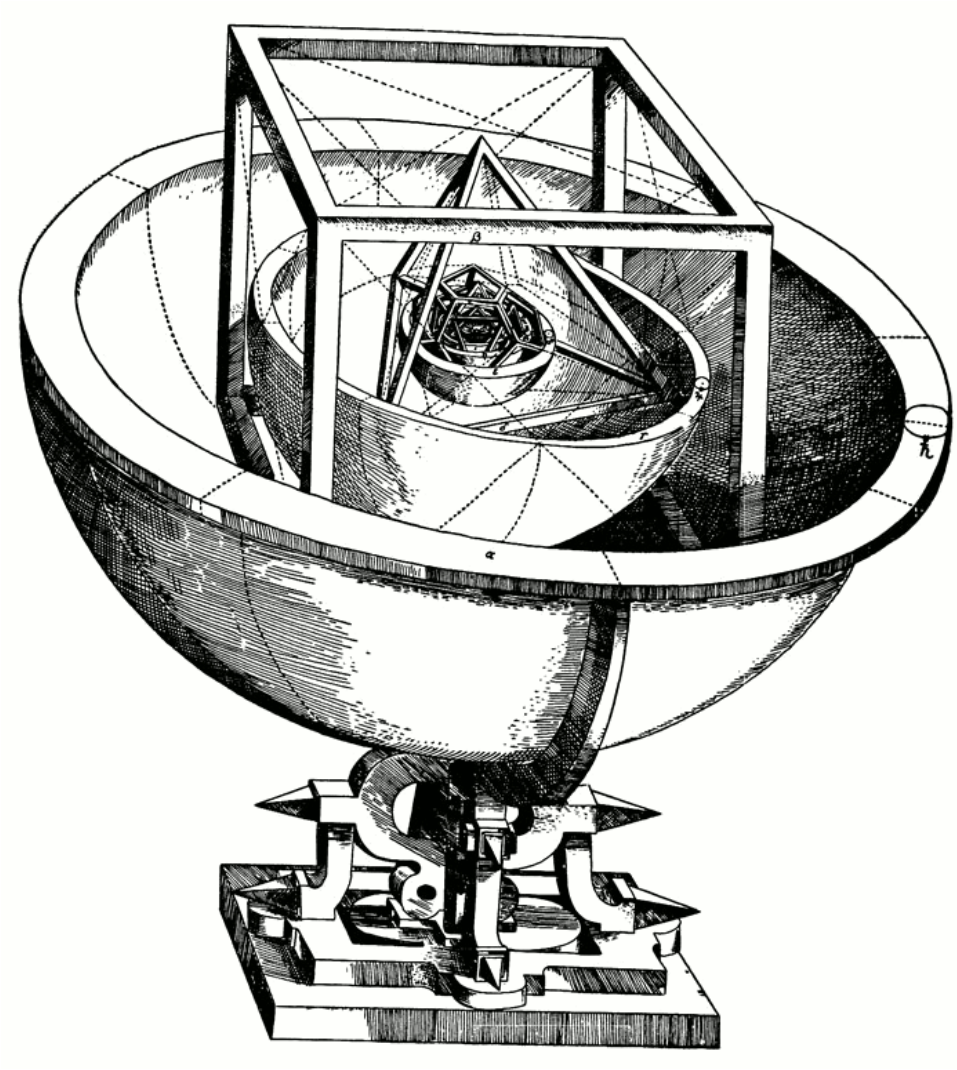


(fire) (earth) (air) (water) (universe)

A Bit More History...

Johannes Kepler, in his *Mysterium Cosmographicum* (1596), described a heliocentric model of the solar system, in which the five Platonic solids separated the six known planets (Mercury, Venus, Earth, Mars, Jupiter, and Saturn).



Why aren't there other Platonic solids?

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If there really are only these five, how can we show that? How can we *prove* that there can be no others?

The Euler Characteristic

Leonhard Euler (born April 15, 1707; Happy Birthday!) developed a way of assigning a number χ to each polyhedron which captures some of its geometric properties.

This number χ is called the Euler Characteristic in his honor.

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The Euler Characteristic χ for the surface a polyhedron with F faces, E edges, and V vertices is:

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$$\chi = F - E + V \quad \text{or} \quad \chi = F + V - E$$

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Suppose a solid has F faces, each face is an p-sided polygon, and q edges meet at each vertex.

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and since each edge connects 2 vertices, and there are q edges per vertex, there must be:

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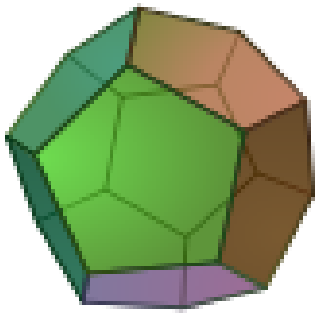
This is nicely summed up with $pF = 2E = qV$.

Euler Characteristic $(F + V - E)$ of Platonic Solids

The Dodecahedron has 12 pentagonal faces (so $p = 5$ edges on each face), and $q = 3$ edges on each vertex.



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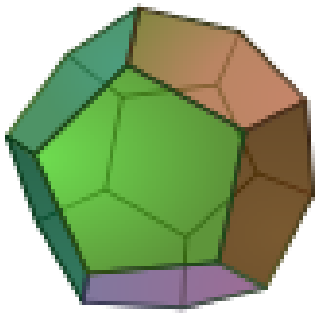

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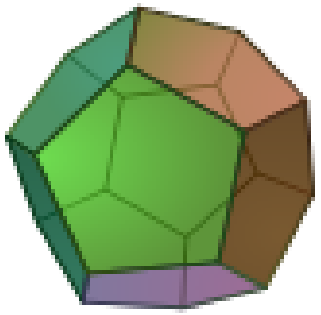

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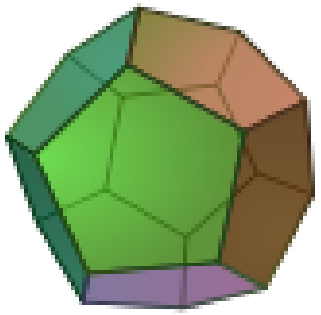

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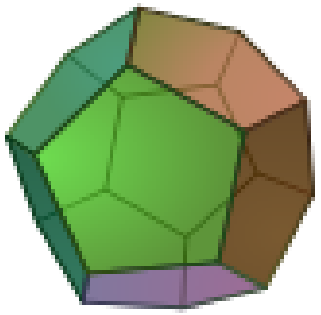

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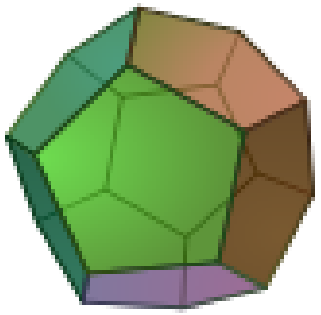

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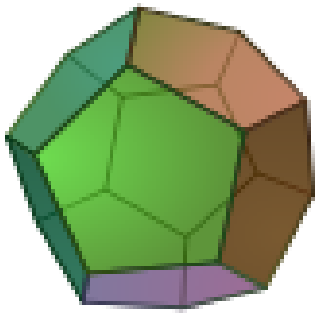

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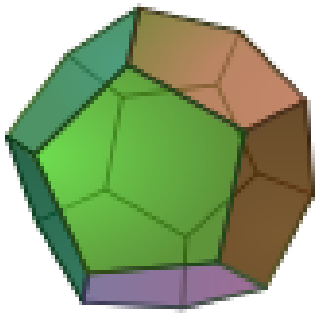

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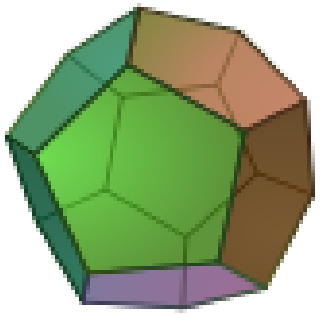

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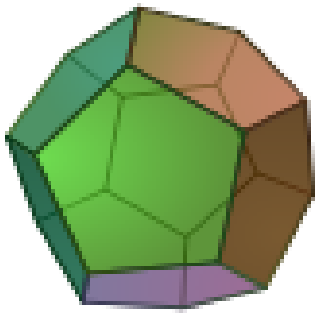

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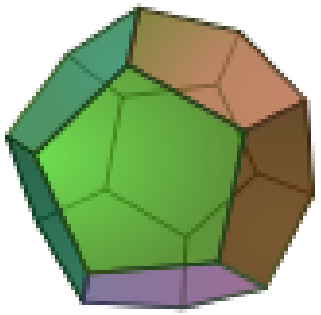

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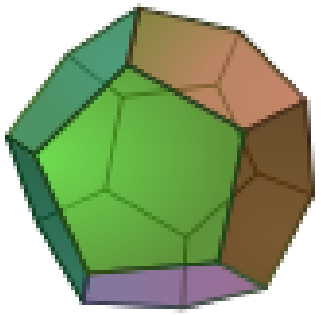

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In all cases, $\chi = 2$! Why?

A Theorem for the Euler Characteristic

For the surface of any convex polyhedron having F faces, E edges, and V vertices, $\chi = F + V - E = 2$.

("Convex" means any line segment joining two points on the polyhedron lies completely inside it.)

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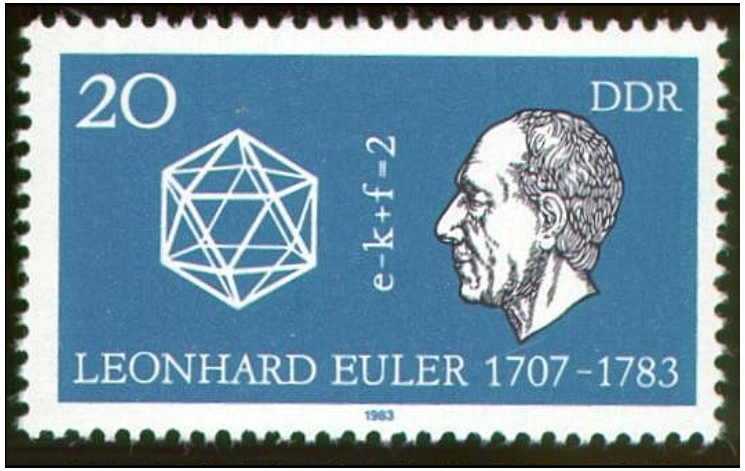
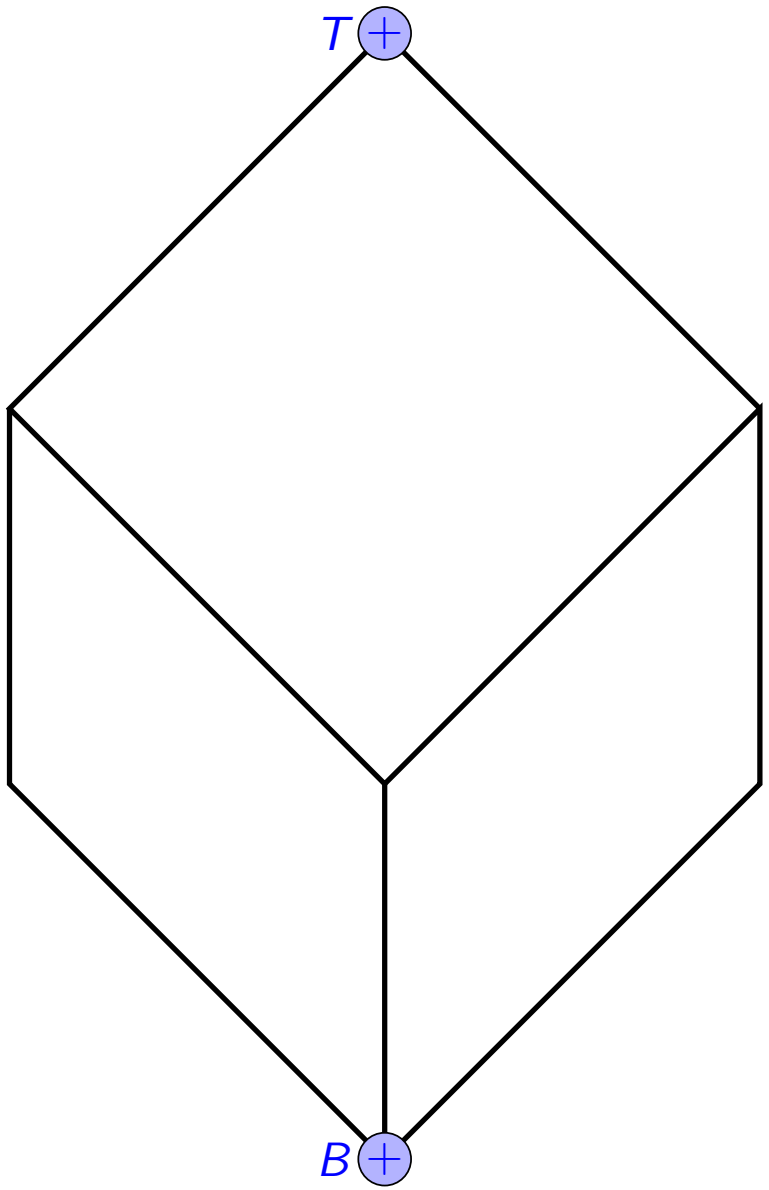


Proving Euler's Polyhedron Formula with Charges

- Position the polyhedron so that no edge is horizontal.
- There is one top vertex T, and one bottom vertex B.



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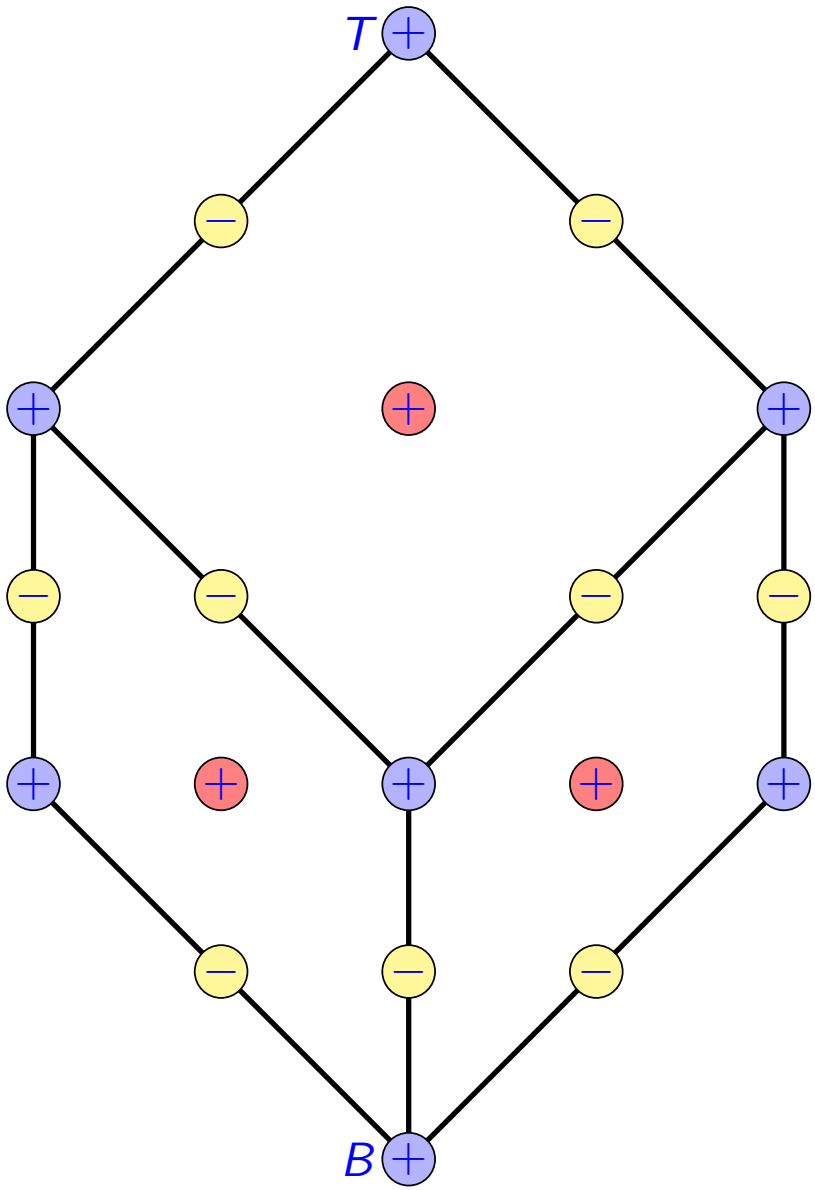

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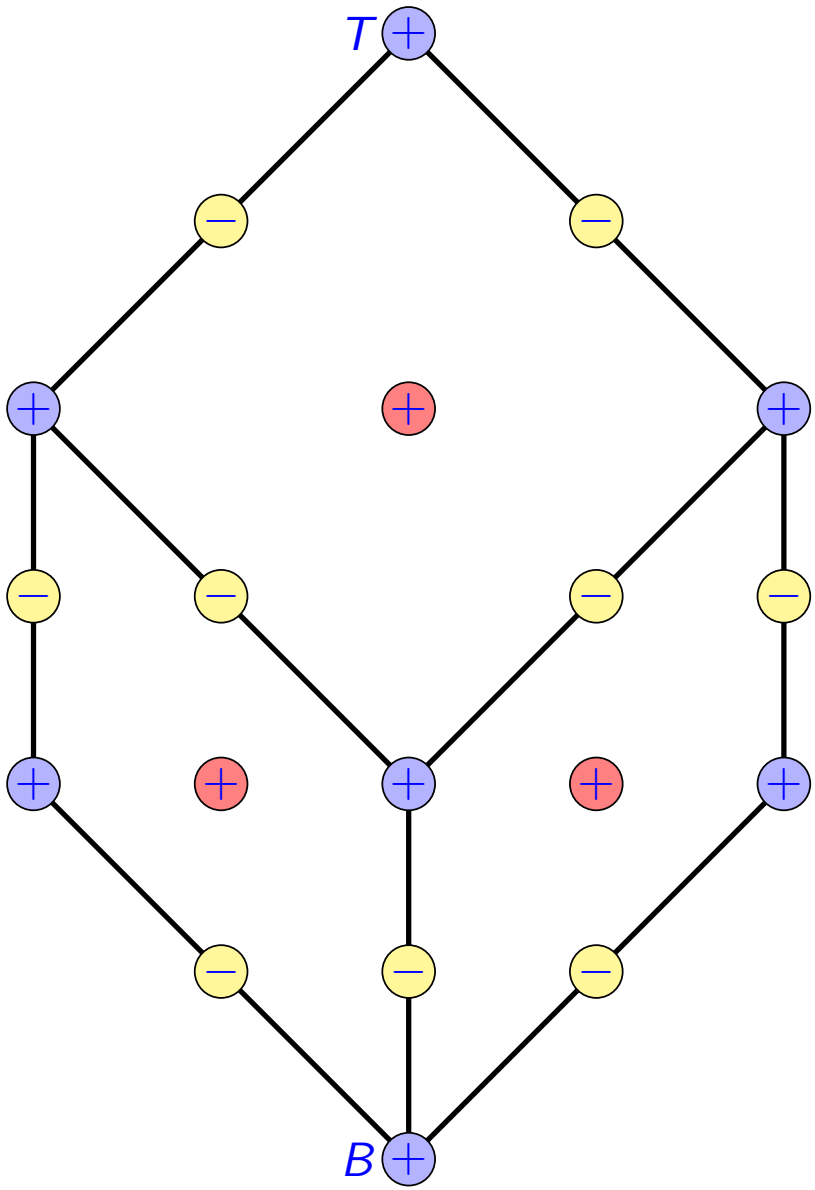

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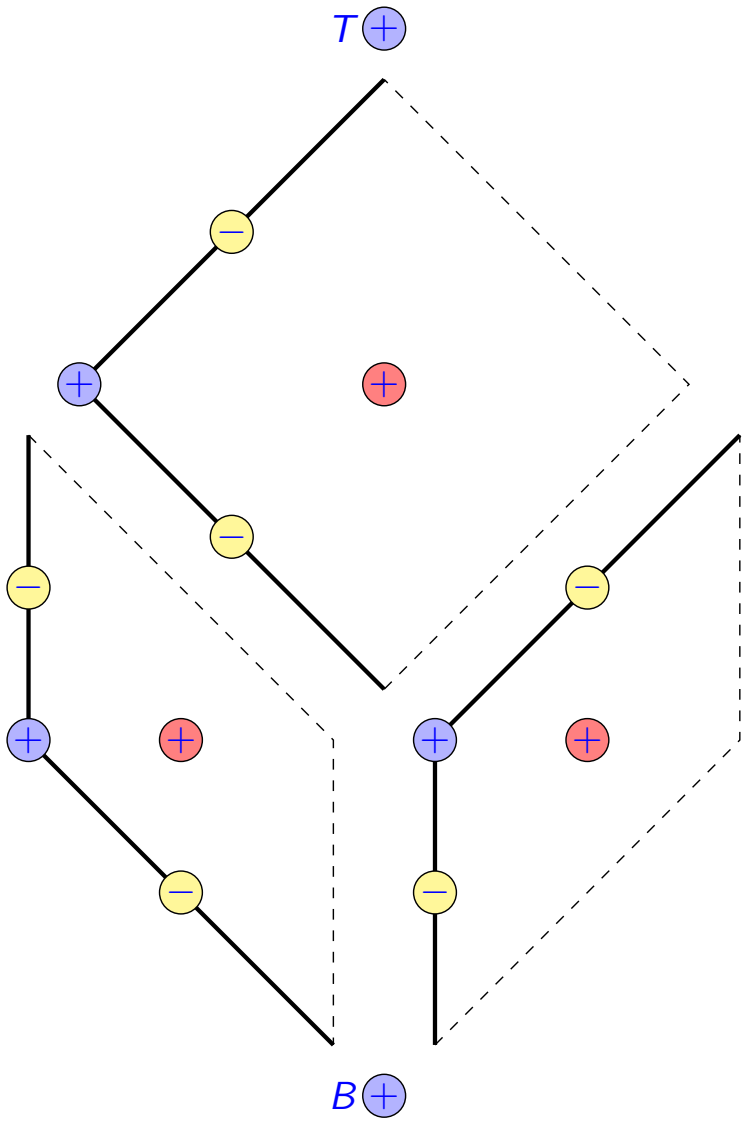
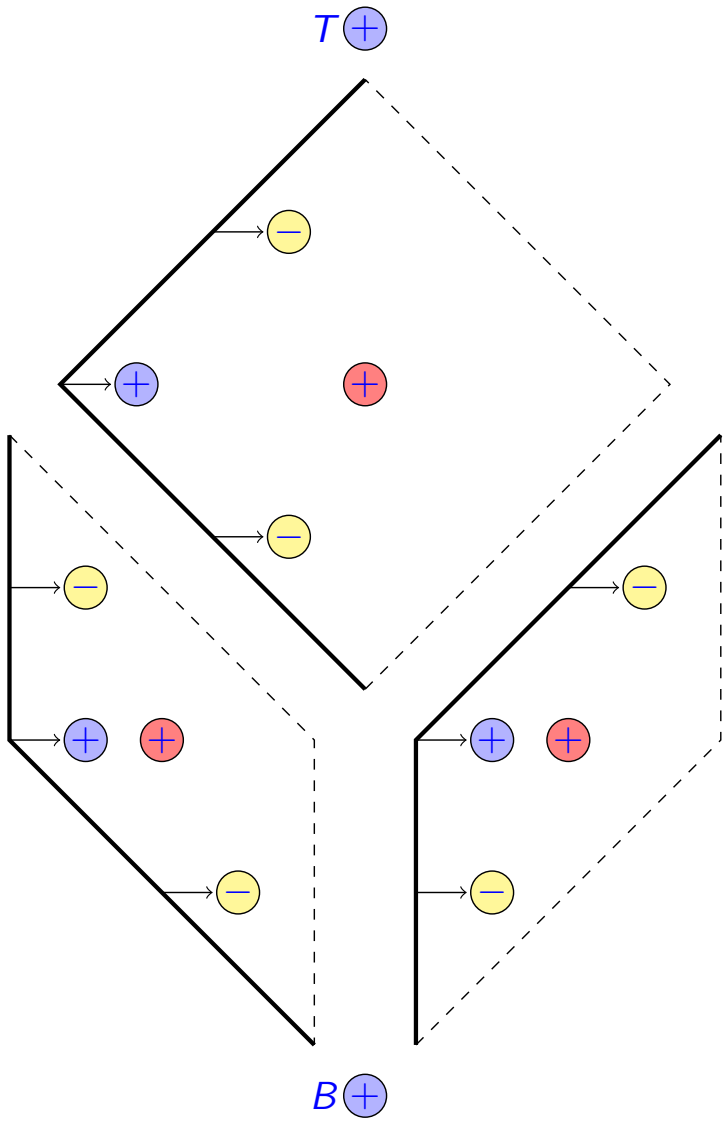


Proving Euler's Polyhedron Formula with Charges

- Position the polyhedron so that no edge is horizontal.
- There is one top vertex T, and one bottom vertex B.
- Place a "$+$" on each vertex or face, and a "$-$" on each edge.
- If a charge has a face immediately counterclockwise (viewed from the top), push it onto that face. (Charges in front go left; charges in back go right.)
- T and V have no face, all others do. On each face, the sum is 0.



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So we have

$$\frac{2E}{p} + \frac{2E}{q} - E = 2 > 0.$$

Continuing the calculation...

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$$\begin{aligned} \frac{2E}{p} + \frac{2E}{q} - E &> 0 \\ \frac{1}{p} + \frac{1}{q} - \frac{1}{2} &> 0 \end{aligned}$$

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Continuing the calculation...

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Now, p and q must both be larger than 2.

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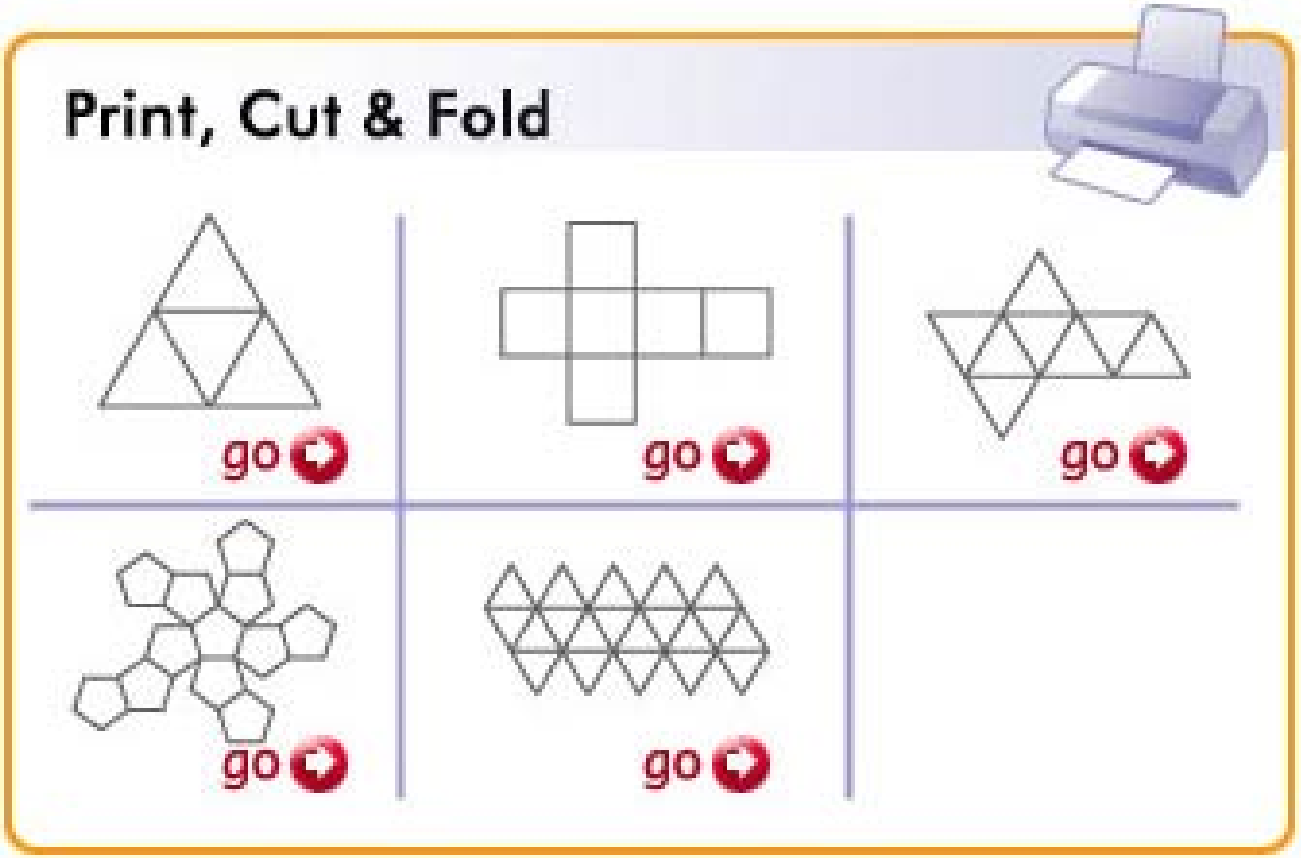
- $p = 3$, $q = 5$ — $\frac{1}{3} + \frac{1}{5}$ — Icosahedron — 

My Favorite Modern Application of Platonic Solids

Because they make great dice, many modern board games use regular polyhedra for dice. So if you want your own set of platonic solids, you have options in many materials, colors, and designs.

Can you spot the non-platonic solid?

Print, Cut, and Fold Your Own Platonic Solids!



Visit `http://www.learner.org/interactives/geometry/platonic.html` For Print and Fold instructions to make your own Platonic Solids!

References

- Frank J. Swetz (The Pennsylvania State University), "Leonardo da Vinci's Geometric Sketches," *Convergence* (June 2010), DOI:10.4169/loci002559
- John Baez. "Who Discovered the Icosahedron?," Retrieved from: http://math.ucr.edu/home/baez/icosahedron/1.html
- George W. Hart. "Johannes Kepler's Polyhedra," Retrieved from: http://www.georgehart.com/virtual-polyhedra/kepler.html
- Twenty Proofs of Euler's Formula. `https://www.ics.uci.edu/~eppstein/junkyard/euler/`
- A ton of great information can be found on WikipediA at:
  - `https://en.wikipedia.org/wiki/Platonic_solid`
  - `https://en.wikipedia.org/wiki/Mysterium_Cosmographicum`
  - `https://en.wikipedia.org/wiki/Euler_characteristic`

Thank you!