Math 173 **Exam 2 Solutions**
Section 7
Fall 2008

2.a. Let a be a number. Express the slope of the tangent line of $y = x + 2/x$ at the point $(a, a + 2/x)$ as a limit. Do **not** evaluate the limit.

Solution: The slope of the tangent line is the derivative, $\frac{dy}{dx}$, of the function (in this case, $y = f(x) = x + 2/x$). By definition, we have

$$\begin{aligned} f'(x) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x + 2/x - (a + 2/a)}{x - a} \end{aligned}$$

b. Evaluate the limit.

Solution: Using part a, we have:

$$\begin{aligned} f'(x) &= \lim_{x \to a} \frac{x + 2/x - (a + 2/a)}{x - a} \\ &= \lim_{x \to a} \frac{x - a + 2/x - 2/a}{x - a} \\ &= \lim_{x \to a} (\frac{x - a}{x - a} + \frac{2/x - 2/a}{x - a} \\ &= \lim_{x \to a} \frac{x - a}{x - a} + \lim_{x \to a} \frac{2/x - 2/a}{x - a} \\ &= \lim_{x \to a} 1 + \lim_{x \to a} \frac{xa(2/x - 2/a)}{xa(x - a)} \\ &= 1 + \lim_{x \to a} \frac{2a - 2x}{xa(x - a)} \\ &= 1 + \lim_{x \to a} \frac{-2(x - a)}{xa(x - a)} \\ &= 1 + \lim_{x \to a} \frac{-2}{xa} \\ &= 1 - 2/a \lim_{x \to a} \frac{1}{x} \\ &= 1 - 2/a^2 \end{aligned}$$

4. Differentiate the function.

a. $f(x) = x^7 + 2x^3 - 2x + 1$

Solution: Use the power rule on each term to obtain $f'(x) = 7x^6 + 6x^2 - 2$.

b. $g(t) = t + \frac{1}{t^2+1}$

Solution: Differentiate each term separately. Use the quotient rule on the second term: $g'(t) = t' + (\frac{1}{t^2+1})' = 1 + \frac{1'(t^2+1) - 1(t^2+1)'}{(t^2+1^2} = 1 + \frac{0(t^2+1)-1(2t)}{(t^2+1)^2} = 1 + \frac{-2t}{(t^2+1)^2}$.

c. $y = x^2 e^x$

Solution: Use the product rule: $y' = (x^2)'e^x + x^2(e^x)' = 2xe^x + x^2e^x$.

d. $c(\theta) = 1 + \cot\theta$

Solution: $c'(\theta) = 1' + (\cot\theta)' = 0 + (\frac{\cos\theta}{\sin\theta})' = \frac{(\cos\theta)'\sin\theta - \cos\theta(\sin\theta)'}{\sin^2\theta} = \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta} = \frac{-1}{\sin^2\theta} = -\csc^2\theta$.

e. $r(x) = e^{-x}\sin x$

Solution: $r'(x) = (\frac{\sin x}{e^x})' = \frac{(\sin x)'e^x - \sin x(e^x)'}{e^{2x}} = \frac{\cos x e^x - \sin x e^x}{e^{2x}} = \frac{\cos x - \sin x}{e^x} = e^{-x}(\cos x - \sin x)$.

5.a. Let a be a number. Find the equation of the tangent line to the curve $y = (2x+5)^3$ at the point $(a, (2a+5)^3)$.

Solution: The equation of a line is $y - b = m(x-a)$, where m is the slope of the line and (a, b) is a point on the line. The tangent line passes through the point $(a, (2a+5)^3)$, and has slope $f'(a)$, where $y = f(x) = (2x+5)^3$. Using the chain rule, we have $f'(x) = (3(2a+5)^2)(2) = 6(2a+5)^2$. Therefore the equation of the tangent line is

$$y - b = m(x-a)y - f(a) = f'(a)(x-a)y - (2a+5)^3 = 6(2a+5)^2(x-a)$$

b. For what values of the number a is the tangent line to the curve $y = (2x+5)^3$ parallel to the line $y = 54x + 7$?

Solution: The slop of the line $y = 54x + 7$ is 54, and the slope of the tangent line is $6(2a+5)^2$. The lines are parallel if and only if their slopes are equal. Therefore we need to solve the follwoing equations for a:

$$\begin{aligned} 6(2a+5)^2 &= 54 \\ (2a+5)^2 &= 9 \\ 2a+5 &= \pm 3 \\ 2a &= -8 \text{ or } -2 \\ a &= -4 \text{ or } -1 \end{aligned}$$

6. Differentiate the function:

a. $f(x) = \sin(x^3 + 8)$

Solution: $f(x) = g \circ h(x)$, where $g(y) = \sin y$ and $h(x) = x^3 + 8$. Thus $f'(x) = g' \circ h(x)h'(x) = \cos(x^3 + 8)3x^2$.

b. $g(x) = \ln(7x^2 + 5)$

Solution: $g(x) = f \circ h(x)$, where $f(y) = \ln y$ and $h(x) = 7x^2 + 5$. Thus $g'(x) = f' \circ h(x)h'(x) = \frac{1}{7x^2+5}14x$.

c. $h(t) = \sqrt{\frac{t}{t^2+4}}$

Solution: $h(t) = f \circ g(t)$, where $f(x) = \sqrt{x}$ and $g(t) = \frac{t}{t^2+4}$. Thus $h'(t) = f' \circ g(t)g'(t) = \frac{1}{2}(\frac{t}{t^2+4})^{-1/2}\frac{t^2-2t+4}{(t^2+4)^2}$.

d. $y = e^{\alpha x} \sin \beta x$

Solution: Use the chain rule to differentiate $f(x) = e^{\alpha x}$: $f(x) = g \circ h(x)$, where $g(t) = e^t$ and $h(x) = \alpha x$, so $f'(x) = g'(h(x))h'(x) = e^{\alpha x}\alpha = \alpha e^{\alpha x}$. Similarly, $\frac{d}{dx}(\sin \beta x) = \beta \cos \beta x$. Now use the product rule on y:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(e^{\alpha x}) \sin \beta x + e^{\alpha x}\frac{d}{dx}(\sin \beta x) \\
&= \alpha e^{\alpha x} \sin \beta x + e^{\alpha x}\beta \cos \beta x \\
&= \alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x
\end{aligned}$$

e. $y = 2x \log_{10} \sqrt{x}$

Solution: First, observe that $\log_{10} \sqrt{x} = \log_{10}(x^{1/2}) = \frac{1}{2}\log_{10} x$, so $y = 2x(\frac{1}{2}\log_{10} x) = x \log_{10} x$. Thus $\frac{dy}{dx} = (x)' \log_{10} x + x(\log_{10} x)' = \log_{10} x + x\frac{1}{(\ln 10)x} = \log_{10} x + \frac{1}{\ln 10}$.

7. The equation $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ describes a curve in the plane.

a. Verify that the point $(-3, 1)$ lies on the curve.

Solution:

$$\begin{aligned}
2(x^2 + y^2)^2 &= 25(x^2 - y^2) \\
2((-3)^2 + 1^2)^2 &= 25((-3)^2 - 1^2) \\
2(9 + 1)^2 &= 25(9 - 1) \\
2 \cdot 100 &= 25 \cdot 8 \\
200 &= 200
\end{aligned}$$

b. Find the equation of the tangent line to the curve at the point $(-3, 1)$.

Solution: The equation of the tangent line at $(-3, 1)$ is $y - b = f'(a)(x - a)$, where $y = f(x)$ is (locally) a function of x, $a = -3$, and $b = 1$. To find $f'(a)$, we first differentiate implicitly, then solve for $\frac{dy}{dx} = f'(a)$. Since we are interested in the value of $\frac{dy}{dx}$ at the point $(-3, 1)$, we can substitute $x = -3$ and $y = 1$ in the equation we get *after* differentiating.

$$\begin{aligned}
2(x^2+y^2)^2 &= 25(x^2-y^2)\\
\frac{d}{dx}[2(x^2+y^2)^2] &= \frac{d}{dx}[25(x^2-y^2)]\\
4(x^2+y^2)(2x+2y\frac{dy}{dx}) &= 50x - 50y\frac{dy}{dx}\\
8y(x^2+y^2)\frac{dy}{dx} + 50y\frac{dy}{dx} &= 50x - 8x(x^2+y^2)\\
(8y(x^2+y^2)+50y)\frac{dy}{dx} &= 50x - 8x(x^2+y^2)\\
(8\cdot 1\cdot((-3)^2+1^2)+50\cdot 1)\frac{dy}{dx} &= 50(-3) - 8(-3)((-3)^2+1^2)\\
(80+50)\frac{dy}{dx} &= -150+240\\
\frac{dy}{dx} &= \frac{9}{13}
\end{aligned}$$

8. Differentiate the function $f(x) = (x+1)^{\sin x}$.
Solution: Use logarithmic differentiation:

$$\begin{aligned}
f(x) &= (x+1)^{sinx}\\
\ln f(x) &= \ln((x+1)^{sinx})\\
\frac{d}{dx}[\ln f(x)] &= \frac{d}{dx}[\ln((x+1)^{\sin x})]\\
\frac{f'(x)}{f(x)} &= \frac{d}{dx}[(\sin x)\ln(x+1)]\\
\frac{f'(x)}{f(x)} &= (\cos x)\ln(x+1) + (\sin x)\frac{1}{x+1}\cdot 1\\
f'(x) &= f(x)(\cos x \ln(x+1) + \frac{\sin x}{x+1})\\
f'(x) &= (x+1)^{sinx}(\cos x \ln(x+1) + \frac{\sin x}{x+1})
\end{aligned}$$