Math 322 **Fall 2008**
Exam 2 Solutions

1.a. List the elements of each conjugacy class of the quaternion group $Q = \{1, -1, i, -i, j, -j, k, -k\}$.
b. For each element $q \in Q$, how many elements of Q commute with q?

Solution to part a: Given $q \in Q$, the conjugacy class of q is $\mathrm{Cl}(q) = \{r \in Q \mid \exists s \in Q, r = sqs^{-1}\}$. Since 1 and -1 are in the center of Q, we have $\mathrm{Cl}(1) = \{1\}$ and $\mathrm{Cl}(-1) = \{-1\}$. We now determine $\mathrm{Cl}(i)$. If $r = 1, -1, i, \text{or} - i$, then $rir^{-1} = i$. If $r = j, -j, k, \text{or} - j$, then $rir^{-1} = -irr^{-1} = -i$. Therefore $\mathrm{Cl}(i) = \{i, -i\}$. Similarly, $\mathrm{Cl}(j) = \{j, -j\}$ and $\mathrm{Cl}(k) = \{k, -k\}$.

Solution to part b: Given $q \in Q$, the centralizer $C_q(Q)$ consists of precisely those elements that commute with q. Since $|C_q(Q)| = |Q|/|\mathrm{Cl}(q)| = 8/|\mathrm{Cl}(q)|$. Using part a, we have $|C_q(Q)| = 4$ if $q \in \{i, -i, j, -j, k, -k\}$ and $|C_q(Q)| = 8$ if $q = \pm 1$.

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2. Let R be the ring of 2×2 matrices with coefficients in $\mathbb{R}$. Prove that the set

$$I = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

is a left ideal of R, but not a right ideal.

Solution: To prove that I is a left ideal, we must show that I is nonempty, closed under subtraction, and closed under multiplication on the left by elements of R. That I is nonempty is clear. We show that I is closed under subtraction:

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} - \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & a-c \\ 0 & b-d \end{pmatrix} \in I,$$

and multiplication:

$$\begin{aligned} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} &= \begin{pmatrix} x \cdot 0 + y \cdot 0 & x \cdot a + y \cdot b \\ z \cdot 0 + w \cdot 0 & z \cdot a + w \cdot b \end{pmatrix} \\ &= \begin{pmatrix} 0 & xa + yb \\ 0 & za + wb \end{pmatrix} \in I \end{aligned}$$

Finally, we can show that I is not a right ideal by demonstrating that it is not closed under multiplication on the right with a counterexample:

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin I$$

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3. Let $\varphi : R \to S$ be a ring homomorphism.
a. Prove that the image of φ is a subring of S.

b. Suppose R is unital. Prove that $\varphi(1)$ is the multiplicative identity of the image of φ.

Solution to part a: To show that $\varphi(R)$ is a subring of S, we must show that it is nonempty and closed under both subtraction and multiplication. It is nonempty, because $0 = \varphi(0) \in S$. To show that it is closed under subtraction, let s_1 and s_2 be elements of $\varphi(R)$. Then there exists r_1 and r_2 of R such that $\varphi(r_1) = s_1$ and $\varphi(r_2) = s_2$. Therefore $s_1 - s_2 = \varphi(r_1) - \varphi(r_2) = \varphi(r_1 - r_2) \in \varphi(R)$.

Solution to part b: To prove that $\varphi(1)$ is the multiplicative identity of $\varphi(R)$, we must show that, given any $s \in \varphi(R)$, $\varphi(1)s = s$ and $s\varphi(1) = s$. Since $s \in \varphi(R)$, there exists $r \in R$ such that $s = \varphi(r)$. Thus $\varphi(1)s = \varphi(1)\varphi(r) = \varphi(1r) = \varphi(r) = s$. $s\varphi(1) = s$ is proved similarly.

4.a. Give an example to show that a subring of a unital ring need not be unital.

b. Give an example of a unital subring of a ring R whose multiplicative identity is not the multiplicative identity of R.

Solution to part a: $2\mathbb{Z}$ is a subring of the unital ring $\mathbb{Z}$. However, $2\mathbb{Z}$ is not unital, because if $i \in \mathbb{Z}$ is such that $i \cdot n = n = 1 \cdot n$ for some nonzero $n \in 2\mathbb{Z}$, then, since $\mathbb{Z}$ is an integral domain, $i = 1$, which is not an element of $2\mathbb{Z}$.

Solution to part b: $R = \{\bar{0}, \bar{3}\}$ is a subring of $\mathbb{Z}_6$, and both are unital. However, the muliplicative identity of R is $\bar{3}$, while the identity of $\mathbb{Z}_6$ is $\bar{1}$.

5.a. Given ideals I and J of a ring R, prove that the set $I + J = \{i + j \mid i \in I, j \in J\}$ is an ideal.

b. Suppose K is an ideal of R such that $I \subseteq K$ and $J \subseteq K$. Prove that $I + J \subseteq K$.

Solution to part a: To prove that $I+J$ is an ideal, we must show that $I+J$ is nonempty, closed under subtraction, and closed under multiplication (on the right and on the left) by elements of R. $I + J$ is nonempty since $0 \in I$ and J, so $0 = 0 + 0 \in I + J$. Suppose $i_1 + j_1$ and $i_2 + j_2$ are elements of $I + J$. Then $(i_1 + j_1) - (i_2 + j_2) = i_1 + j_1 - i_2 - j_2 = i_1 - i_2 + j_1 - j_2 \in I + J$.

To show that $I+J$ is closed under multplication by elements of R, let $r \in R$ and let $i+j \in I+J$. Then $r(i + j) = ri + rj \in I + J$, since I and J are closed under multiplication. Therefore $I + J$ is closed under multplication on the left by elements of R. Similarly, $I + J$ is closed under multiplication on the right by elements of R.

Solution to part b: Let $i + j \in I + J$. Since $i \in I$, $i \in K$. Similarly $j \in K$. Since K is closed under addition, $i + j \in K$.

6.a. Let $R = \{0, 1, a, b\}$ be a commutative ring such that $R^\times = \{1, a, b\}$. Explain why R must be an integral domain. What is the characteristic of R?

b. Make addition and multiplication tables for R.

Solution to part a: Every nonzero element of R is invertible, hence not a zerodivisor: If x is invertible and $xy = 0$, then $y = x^{-1}xy = x^{-1}0 = 0$. The characteristic of R must be a prime divisor of $4 = |R|$, hence it is 2.

Solution to part b: Since the characteristic is 2, we have $1 + 1 = a + a = b + b = 0$. If $1 + a = 0$, then $1 + 1 + a = 1 + 0$, which implies $a = 1$, a contradiction. If $1 + a = 1$ then $a = 0$, which is also a contradiction. If $1 + a = a$ then $1 = 0$, contradiction. Therefore $1 + a = b$. We now have sufficient information to complete the addition table for R:

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

The multiplication table for R can be filled in once we observe that $0 \cdot r = r \cdot 0 = 0$ for all $r \in R$, and that $R^\times$ is a mulitplicative group of order 3, for which there is only one possibility (up to isomorphism).

·	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

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7. Let R be an integral domain of characterstic $p > 0$, and let $R' = \{k \cdot 1 \mid k \in \mathbb{Z}\}$ be the cyclic subgroup of R generated by 1. Prove that R' is a subring of R.

Solution: Since R' is, by definition, an additive subgroup of R, we need only to prove that R' is closed under multiplication. Let r and s be elements of R'. Then there exist integers l and k such that $r = k \cdot 1$ and $s = l \cdot 1$. Then $rs = (k \cdot 1)(l \cdot 1) = kl \cdot 1 \in R'$.

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8. Let F be a field and $I \subseteq F$ an ideal. Prove that if $I \neq \{0\}$, then $I = F$.

Solution: Since $I \neq \{0\}$, there exists a nonzero element $i \in I$. Let a be an arbitrary element of F. Since F is a field and $i \neq 0$, i has a multiplicative inverse $i^{-1} \in F$. Since I is closed under multiplication by elements of F, $a = a \cdot 1 = a(i^{-1}i) = (ai^{-1})i \in I$. Since $a \in F$ is arbitrary, this shows that $F \subseteq I$. Since the reverse inclusion $I \subseteq F$ clearly holds, we conclude that $I = F$.