# 1 Chpater 5: Integrals

For Problems 1-4, refer to the following table:

| | |
|---|---|
| $f(x) = x^2 - 3x,\ a = 3,\ b = 4$ | $f(x) = 3\csc^2 x,\ a = \frac{\pi}{6},\ b = \frac{\pi}{2}$ |
| $f(x) = 1 - 2\cos x,\ a = 0,\ b = 1$ | $f(x) = \frac{x^2+x+1}{\sqrt{x^3}},\ a = 1,\ b = 2$ |
| $f(x) = \frac{1}{4+4x^2},\ a = 1,\ b = \frac{1}{2}$ | $f(x) = \frac{2x}{4+4x^2},\ a = -1,\ b = 2$ |
| $f(x) = \frac{\sin^7 x}{\sec x},\ a =,\ b =$ | $f(x) = (x+5)\sqrt{x+2},\ a = 3,\ b = -5$ |
| $f(x) = \frac{\sin x}{1+\cos^2 x},\ a = -\pi/2,\ b = 0$ | $f(x) = \csc 2x \cot 2x,\ a = \pi/4,\ b = 3\pi/4$ |

**1.** Express the net area between the graph of the given function and the $x$-axis over the given interval as a limit. Use Riemann sums with left endpoints.

**2.** Approximate the area between the graph of the given function and the $x$-axis with a Riemann sum with 2, 3, and 4 subintervals, using right endpoints.

**3.** Compute the definite integral $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus.

**4.** Compute the given indefinite integral using the Fundamental Theorem of Calculus.

**5.** Express the net area between the graph of the function $f(x) = \cos 2x$ and the $x$-axis over the interval $[3, 5]$ as a limit. Use Riemann sums with **right** endpoints.

**6.** Determine a region whose area is equal to the given limit.

**a.** $\lim_{n\to\infty} \sum_{i=1}^n \frac{\pi}{4n} \sin(\frac{i\pi}{4n})$.

**b.** $\lim_{n\to\infty} \sum_{i=1}^n \frac{\pi}{2n} \sin(\frac{i\pi}{4n})$.

**c.** $\lim_{n\to\infty} \sum_{i=1}^n \frac{\pi}{4n} \sin(\frac{i\pi}{2n})$.

**d.** $\lim_{n\to\infty} \sum_{i=1}^n \frac{\ln(\frac{7i}{n})}{i}$.

**7.** Compute the given integral using the definition (*i.e.*, as a limit of Riemann sums). You may use the properties of integrals, such as $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. The following identities may be useful:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

**a.** $\int_0^1 (x+1)\,dx$

**b.** $\int_1^3 (x^2 + 2x - 1)\,dx$

**8.** **a.** Determine a region whose area is equal to the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i^2}{n^2}+\frac{i}{n}+1\right)\frac{1}{n}$$

**b.** Directly evaluate the limit in part a.

**9.** Approximate the area between the graph of the function $f(x) = x^2 + 1$ and the $x$-axis over the interval $[2, 3]$ with a Riemann sum with 4 subintervals, using **left** endpoints.

**10.** Compute the definite integral using the Fundamental Theorem of Calculus.

**a.** $\int_0^{\pi/2}(4x^3 + \sin x)\,dx$

**b.** $\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$

**c.** $\int_1^e \frac{\ln|x|}{2x}\,dx$

**d.** $\int_4^9 \frac{5x^2+2\sqrt{x}}{x}\,dx$

**11.** Compute the given indefinite integral using the Fundamental Theorem of Calculus.

**a.** $\int(1 - \frac{2x}{\sqrt{1-x^2}})\,dx$

**b.** $\int \frac{5+3\cos 2x - x^7}{2}\,dx$

**c.** $\int \tan x\,dx$

**d.** $\int e^{2x+7}\,dx$

**12.** State both parts of the Fundamental Theorem of Calculus.

**13.** Find the derivative of the given function.

**a.** $F(x) = \int_{-5415}^{x} \sin^6(3t)\,dt$.

**b.** $F(x) = \int_x^0 \sin(3t)\,dt$.

**c.** $F(x) = \int_0^{-x} \cos^6(t)\,dt$.

**d.** $F(x) = \int_0^{x^2} \frac{t}{1+t^2}\,dt$.

**14.** You begin driving down a highway at noon. You are initially at rest. Your acceleration is given by the function $a(t) = t^2 + 10t$, where the time $t$ is measured in hours and the distance from your starting point is measured in kilometers.

**a.** Find the velocity $v(t)$ as a function of time.

**b.** Find the distance you have travelled $s(t)$ as a function of time.

**15.** Suppose you are filling an empty drum with a hose. The rate at which water is coming out of the hose at time $t$ is $r(t) = 4 - \frac{\sqrt{t}}{10}$ liters per second, where $0 \le t \le 1600$. Find the total amount of water in the drum after one minute.

**16.** If $f$ is continuous and $\int_0^{125} f(x)\,dx = 9$, find $\int_0^5 x^2 f(x^3)\,dx$.

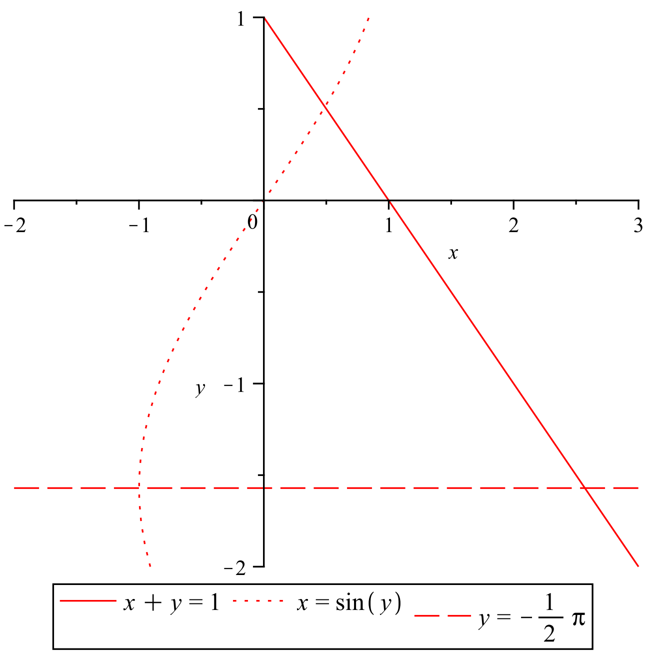
**17.** Find the **derivative** of the function $F(x) = \int_x^1 \frac{1}{1+t^2}\,dt$.

**18.** Suppose you are filling an empty drum with a hose. The rate at which water is coming out of the hose at time $t$ is $r(t) = 4t - \sqrt{t}$ liters per second. Find the total amount of water in the drum after the first $25$ seconds.

**19.** If $f$ is continuous and $\int_0^{100} f(x)\,dx = 12$, find $\int_0^{10} 3xf(x^2)\,dx$.

# 2 Chpater 6: More Integrals

**1.** Express the area of the region bounded by the given curves as an integral (with respect to either $x$ or $y$). Then find the area.




**2.** Consider the triangular region with vertices $(1, 2)$, $(1, 0)$, and $(4, 5)$. Find the volume of the solid obtained by rotating this region about the $x$-axis.

**3.** In each part, sketch the region bounded by the given curves. Express the area of the region as an integral (with respect to either $x$ or $y$). Then find the area.

**a.** $y = 1/x$, $y = 2/x^2$, $x = 1$.

**b.** $y = \sin^2 x$, $y = -\cos^2 x$, $x = 0$, $x = 1$.

**c.** $y = \tan x$, $y = 2\sin x$, $\pi/3 \le x \le \pi/3$.

**d.** $4x + y^2 = 12$, $x = y$.

**e.** $y = x^2$, $y = \frac{2}{x^2+1}$.

**4.** Find the area of the triangle with vertices $(0,0)$, $(1,0)$, and $(4,7)$.

**5.** **a.** Find the volume of the solid obtained by rotating the triangle with vertices $(0,0)$, $(1,0)$, and $(4,7)$ about the $x$-axis.

**b.** Same question, but rotate about the $y$-axis.

**6.** In each part, sketch the region bounded by the given curves. Express the volume of the solid obtained by rotating the region about the given line as an integral (with respect to either $x$ or $y$). Then find the volume.

**a.** $y = 1/x$, $y = 2/x^2$, $x = 1$, about the $x$-axis.

**b.** $y = 1/x$, $y = 2/x^2$, $x = 1$, about the $y$-axis.

**c.** $y = x^2$, $x = y^2$, about the $x$-axis.

**d.** $y = x^2$, $x = y^2$, about the $y$-axis.

**e.** $y = x^2$, $x = y^2$, about the line $x = 2$.

**f.** $4x + y^2 = 12$, $x = y$, about the line $x = 5$.

**g.** $4x + y^2 = 12$, $x = y$, about the line $y = 5$.

**7.** Suppose it takes 3 J of work to stretch a spring from its natural length of $10$ cm to a length of $15$ cm. How much work is required to stretch the spring from:

**a.** a length of $10$ cm to a length of $18$ cm?

**b.** a length of $25$ cm to a length of $30$ cm?

**8.** How much work is done in lifting a $5$ lb Calculus textbook $2$ ft off a table?

**9.** A chain is hanging from a building and has a (uniformly distributed) mass of $100$ kg. How much work is required to lift the chain to the top of the building?

**10.** The amount (in grams) of carbon-14 present in a certain sample at time $t$ (measured in years after 624 BC) is

$$m(t) = 17e^{-0.06t}$$

Find the average amount of carbon-14 present in the sample during the period from 624 BC to 546 BC.

**11.** An object falls under the influence of gravity with a constant acceleration fo $9.8$ m/s$^2$. Suppose the object begins at rest at a height $h$ meters above the ground.

**a.** What is the average velocity of the object in the interval from $t = 0$ to $t = 10$?

**b.** What is the average height of the object in the interval from $t = 0$ to $t = 10$?

**12.** At a certain location the temperature (in °C) in hours after 7 AM was modeled by the function

$$T(t) = 2\sin\frac{\pi t}{15} - 7$$

Find the average temperature over the period from 7 AM to 7 PM.

**13.** Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the given axis.

**a.** $x = 1 + y^2$, $x = 0$, $y = 1$, $y = 2$, about the $x$-axis.

**b.** $\sin x^2 = y$, $y = 0$, $x = \frac{\pi}{2}$, about the $x$-axis.

**c.** $y = x - x^2$, $y = 0$, about $x = 2$, about the line $x = 2$.

# 3 Chpater 7: Techniques of Integration

**1.** In each part, evaluate the given indefinite integral.

**a.**

$$\int \sec^4 x \; dx$$

**b.**

$$\int x \sin^3 x \; dx$$

**c.**

$$\int x\sqrt{x^2 - 9} \; dx$$

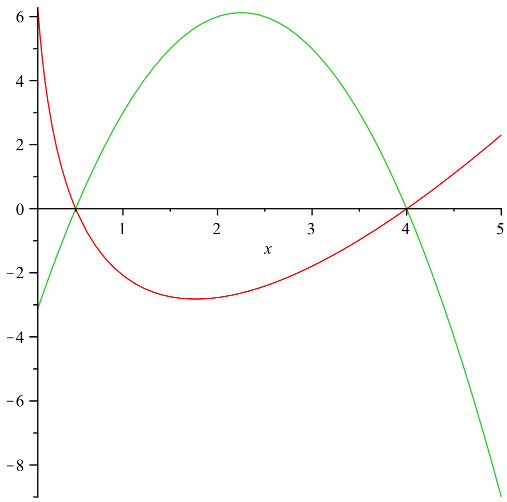
**d.**

$$\int \sqrt{x^2 - 9} \; dx$$

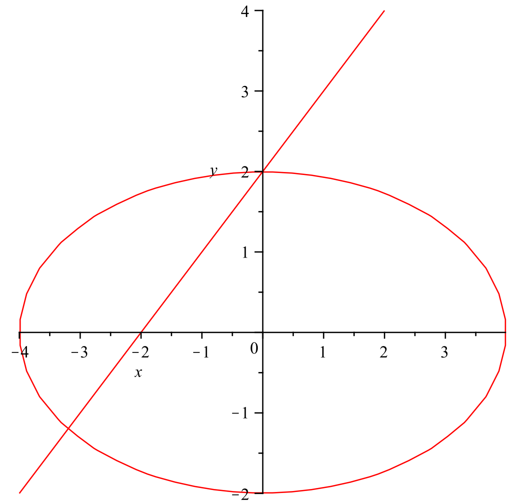
**2.** Do as many of the following exercises as you can from section 7.5: 1 - 10, 13 - 24, 27 - 30, 33 - 47, 54, 57 - 61.

**3.** Find the area of the region bounded by the curves $y = xe^{x/2}$, $y = 0$, and $y = 7$.

**4.** Find the area of the region bounded by the curves $y = (x-4)\ln 2x$ and $y = -2x^2+9x-4$.



Problem 3



Problem 8

**5.** Suppose that $f(1) = 3$, $f(2) = 2$, $f'(1) = -4$, and $f''$ is continuous. Find the value of $\int_1^2 x f''(x)\, dx$.

**6.** Find the area obtained by rotating the region bounded by the curves $y = \sin x$, $y = 1 + \cos x$, $x = 0$, and $x = \pi/2$ about the line $y = -1$.

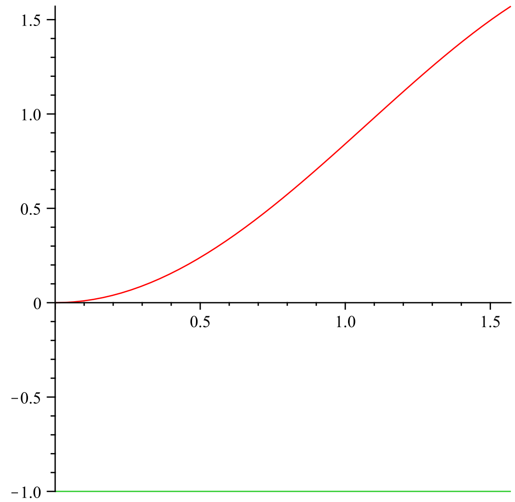
**7.** Find the area obtained by rotating the region bounded by the curves $y = 0$, $y = \sin x + \cos x$, $x = 0$, and $x = \pi/2$ about the line $y = -1$.

**8.** Find the area obtained by rotating the region bounded by the curves $y = \sin x$, $y = \sin x + \cos x$, $x = 0$, and $x = \pi/2$ about the line $y = -1$.

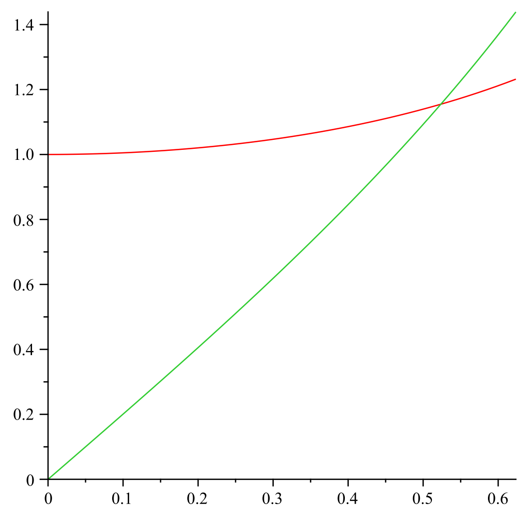
**9.** The line $x = y + 1$ divides the ellipse $x^2 + 4y^2 = 1$ into two parts. Find the area of both parts.

**10.** Find the volume obtained by rotating the circle $x^2 + (y - 2)^2 = 1$ about the $y$-axis.

**11.** Find the area of the region bounded by the curves $y = x \sin x$, $y = -1$, $x = 0$, and $x = \frac{\pi}{2}$.




**12.** Suppose that $f(2) = 2$, $f(-3) = 0$, $f$ is continuous, and the area between the graph of $f$ and the $x$-axis over the interval between $x = -3$ and $x = 2$ is 5. Find the value of $\int_{-3}^{2} x f'(x)\ dx$.




**13.** Find the area obtained by rotating the region bounded by the curves $y = \sec x$, $y = 2\tan x$, $x = 0$, and $x = \pi/6$ about the $x$-axis.

**14.** The line $x = 1$ divides the circle $x^2 + y^2 = 4$ into two parts. Find the area of both parts.

**15.** Find the volume obtained by rotating the circle $x^2 + y^2 = 1$ about the line $y = 3$.

**16.** Compute the improper integral $\int_{-\infty}^{0} xe^x\ dx$.

**17.** Section 7.8, Exercises 5 - 15 odd.

# 4 Chapter 9: Differential Equations

**1.** Verify that the function $y = 2e^{5x}$ is a solution of the initial value problem $\frac{dy}{dx} = 4y$, $y(0) = 2$.

**2.** Verify that, for any constants $C$ and $D$, the function $y = Ce^x + De^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$.

**3.** Verify that, for any constant $C$, the function $y = Ce^{x^2/2}$ is a solution of the differential equation $\frac{dy}{dx} = xy$.

# 5 Chapter 10: Parametric equations and polar coordinates

**1.** Find the implicit ("Cartesian") equation of the given parametrized curve.

**a.** $x = 2 + 3t$, $y = t + 1$.

**b.** $x = t^2$, $y = t + 1$.

**c.** $x = \cos\theta$, $y = \sin\theta$.

**d.** $x = \sec\theta$, $y = \tan\theta$.

**e.** $x = \ln t$, $y = \sqrt{t}$.

**2.** Check that the given point lies on the parametrized curve. Then find the equation of the tangent line to the curve at the given point.

**a.** $x = 2 + 3t$, $y = t + 1$.

**b.** $x = t^2$, $y = t + 1$.

**c.** $x = \cos\theta$, $y = \sin\theta$.

**d.** $x = \sec\theta$, $y = \tan\theta$.

**e.** $x = \ln t$, $y = \sqrt{t}$.

**3.** Find the area enclosed by the given curves.

**a.** The curve $x = t^2$, $y = \sqrt{t}$ $(t \geq 0)$, and the $y$-axis.

**b.** The curve $x = \cos^3 t$, $y = \sin^3 t$, the $x$-axis, and the $y$-axis.

**c.** The curve $x = 1 + e^t$, $y = t - t^2$ $(t \geq 0)$, and the $x$-axis.

**4.** Find the length of the given arc.

**a.** $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.

**b.** $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

**c.** $x = t + \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$.

# 6 Chapter 11: Sequences and series

**1.** Determine whether the sequence is convergent. If so, find the limit.

**a.** $a_n = 1 - (0.9)^n$

**b.** $a_n = \frac{3n^2+27n+12}{5n^2+1}$

**c.** $a_n = \tan\left(\frac{\pi n}{n+1}\right)$

**d.** $a_n = \frac{1}{\pi}\tan^{-1}\left(\frac{n^2}{n+1}\right)$

**e.** $a_n = \frac{e^n+e^{-n}}{e^{2n}-1}$

**2.** Determine if the sequence is increasing, decreasing, or not monotone. Also, determine if it is bounded.

**a.** $a_n = \frac{1}{3n+5}$

**b.** $a_n = \frac{n^2}{3n+5}$

**c.** $a_n = \frac{(-1)^n}{3n+5}$

**3.** Determine the first 4 partial sums of the given series.

**a.** $\sum_{n=1}^{\infty} \frac{1}{3n+5}$

**b.** $\sum_{n=1}^{\infty} \frac{n^2}{3n+5}$

**c.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5}$

**4.** Determine if the given series is convergent or divergent.

**a.** $\sum_{n=1}^{\infty} \frac{1}{3n+5}$

**b.** $\sum_{n=1}^{\infty} \frac{n^2}{3n+5}$

**c.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5}$

**5.** Determine whether the given geometric series is convergent. If so find its value.

**a.** $\sum_{n=1}^{\infty} 3^{n-1}$

**b.** $\sum_{n=1}^{\infty} 3^{1-n}$

**c.** $\sum_{n=1}^{\infty} 9^n 10^{-2n}$

**6.** Use the integral test to determine if the series is convergent or divergent.

**a.** $\sum_{n=1}^{\infty} \frac{1}{3n+5}$

**b.** $\sum_{n=1}^{\infty} \frac{1}{3n^2+5}$

**c.** $\sum_{n=1}^{\infty} \frac{n}{e^n}$

**7.** Determine whether the given series converges.

**a.** $\sum_{n=1}^{\infty} \frac{n}{3n+5}$

**b.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2+5}$

**c.** $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{3n^2+5}$

**d.** $\sum_{n=1}^{\infty} \frac{1}{n4^n}$

**e.** $\sum_{n=1}^{\infty} \frac{\cos^4 n}{4^n}$

**f.** $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

**a.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5}$

**b.** $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{3n+5}$

**c.** $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{\sqrt{3n^4+5}}$

**d.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

**e.** $\sum_{n=1}^{\infty} \frac{3n(-1)^{n-1}}{n!}$

**f.** $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{3^n}$

**8.** Determine whether the given series is absolutely convergent, conditionally convergent, or divergent.

**a.** $\sum_{n=1}^{\infty} \frac{1}{n!}$

**b.** $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

**c.** $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

**d.** $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

**a.** $\sum_{n=1}^{\infty} n \left(\frac{2}{3}\right)^n$

**b.** $\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{3}\right)^n$

**c.** $\sum_{n=1}^{\infty} \frac{1}{n^n}$

**d.** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^n}$