1 Definitions

1. Some question will ask you to statea a definition and/or give an example of a defined term. The terms are:

subset	proper subset	intersection	union
power set	cartesian product	relation	map/function
injective map	surjective map	projection map	binary operation
commutative operartion	associative operation	transitive relation	reflexive relation
symmetric relation	antisymmetric relation	partial order	equivalence relation
partition of a set	equivalence class		

2 Chapter 1 - Principles of Counting

1. Suppose that 5 cards are draw from a standard deck, without regard to order. In how many ways can the following hands be drawn?

a. Five $\clubsuit$s.

b. Five cards of the same suit.

c. Three of a kind.

d. Full house with 2 kings.

2. Find the coefficient of x^2y^3 in each of the following expansions.

a. $(2x - y)^5$.

b. $(x + y + 2)^7$

3. Find the coefficient of w^2xy^3z in each of the following expansions.

a. $(u + v + w + x + y + z)^7$.

b. $(5w + x + y - 2z + 3)^{10}$

4. How many terms appear in each of the following expansions?

a. $(2x - y)^5$.

b. $(w + x + y + z)^7$.

c. $(u + v + w + x + y + z)^7$.

5. In each part, instances of the same letter should be treated as identical.

a. How many arrangements of the letters in the word "APPALACHIN" are there?

b. How many arrangements of "APPALACHIN" have no consecutive As?

6. **a.** How many ways are there to distribute 6 quarters, 4 dimes, and 3 nickels to 4 people?

b. How many ways are there to distribute coins as above so that each person get at least one quarter?

7. Find the number of integer solutions to the given equation or inequality, such that each variable is in the specified range.

a. $x_1 + x_2 + x_3 + x_4 = 10$, $0 \leq x_i \leq 10$ for $i = 1, \dots, 4$.

b. $x_1 + x_2 + x_3 + x_4 = 10$, $1 \leq x_i \leq 7$ for $i = 1, \dots, 4$.

c. $x_1 + x_2 + x_3 + x_4 < 10$, $0 \leq x_i \leq 10$ for $i = 1, \dots, 4$.

8. How many integer solutions are there to the pair of equations $x_1 + x_2 = 5$ and $x_1 + x_2 + \cdots + x_7 = 16$, where $x_i \geq 0$ for all $i = 1, \dots, 7$?

9. Suppose you want to walk from the corner of 1^{st} Ave and 46^{th} St to the corner of Park Ave and 55^{th} St. You can do this by walking a total of 4 blocks west and 9 blocks north, in some order. How many ways are there to complete this task?

3 Chapter 2 - Logic

1. Construct truth tables for the following statements. Determine which are a tautology.

a. $\neg p \vee q$, $q \wedge \neg p$, and $\neg q \to \neg p$.

b. $\neg p \vee (q \to (p \vee \neg q))$.

c. $(p \to q) \leftrightarrow (\neg q \to \neg p))$.

d. $[(p \wedge q) \to r] \leftrightarrow [p \to (q \to r)]$.

2. Use the laws of logic to simplify the statement $(p \vee q \vee \neg r) \wedge \neg(r \wedge s \wedge \neg p)$.

3. Negate the following statements and simplify.

a. $p \to q$

b. $p \leftrightarrow q$

c. $(p \vee \neg q) \wedge (\neg p \vee r)$

d. $p \veebar (q \wedge r)$

e. $p \leftrightarrow (q \vee r)$

4. Decide if each of the following arguments is valid. If not, demonstrate this by assigning truth values to each simple statement (that is, provide a counterexample). If so, use the laws of logic and the rules of inference to show that it is valid.

a.
$$\begin{array}{l} p \vee q \vee r \\ p \to q \\ \underline{\neg q \qquad\quad} \\ \therefore r \end{array}$$

b.
$$\begin{array}{l} (p \vee q) \to r \\ s \to p \\ \neg q \to s \\ s \\ \hline \therefore r \end{array}$$

c.
$$\begin{array}{l} (p \vee q) \to r \\ s \to p \\ \neg q \to s \\ \neg s \\ \hline \therefore \neg r \end{array}$$

d.
$$\begin{array}{l} p \to q \\ q \to r \\ \neg s \wedge p \\ \hline \therefore r \end{array}$$

e.
$$\begin{array}{l} p \\ p \to r \\ p \to (q \vee \neg r) \\ \neg q \vee \neg s \\ \hline \therefore s \end{array}$$

f.
$$\begin{array}{l} q \vee p \\ p \to r \\ r \to s \\ \hline \therefore s \vee q \end{array}$$

5. Let $p(x)$, (x), $r(x)$, and $s(x, y)$ denote the following open statements:

$p(x)$:	x is divisible by 2.
$q(x)$:	x is divisible by 3.
$r(x)$:	x is divisible by 4.
$s(x, y)$:	x is divisible by y.

Write the following statements symbolically, and determine if it is true or false.

a. "Every integer is divisible by 2 or 3."

b. "Every integer that is divisible by 4 is divisible by 2."

c. "If an x is divisible by y and y is divisible by 2, then x is divisible by 2."

d. "Some integers are not divisible by 2 nor 3".

6. Determine the truth of each statement, where the universe is the set of all real numbers.

a. $\forall x \exists y [xy = 1]$.

b. $\exists x \exists y [xy = 1]$.

c. $\forall x \forall y [\sqrt{x + y} = \sqrt{x} + \sqrt{y}]$.

d. $\forall x [\sqrt{x^2} = x]$.

7. Negate each statement in the previous two problems.

8. For any universe and any open statements $p(x)$ and $q(x)$, prove the following.

a. $\forall x[p(x) \wedge q(x)] \Rightarrow (\forall x p(x)) \wedge (\forall x q(x))$

b. $(\forall x p(x)) \wedge (\forall x q(x)) \Rightarrow \forall x[p(x) \wedge q(x)]$

c. $\exists x[p(x) \wedge q(x)] \Rightarrow (\exists x p(x)) \wedge (\exists x q(x))$

d. $\exists x[p(x) \to q(x)] \Rightarrow (\neg \forall x p(x)) \vee (\exists x q(x))$

9. Justify the steps in the following argument.

$\forall x[e(x,x)]$
$\forall x \forall y[e(x,y) \to e(y,x)]$
$\forall x \forall y \forall z([e(x,y) \wedge e(y,z)] \to e(x,z))$
$e(a,b) \wedge \neg e(b,c)$

$\therefore \neg e(a,c)$

1. $e(a,b) \wedge \neg e(b,c)$
2. $[e(b,a) \wedge e(a,c)] \to e(b,c)$
3. $\neg e(b,c) \to \neg[e(b,a) \wedge e(a,c)$
4. $\neg e(b,c)$
5. $\neg[e(b,a) \wedge e(a,c)]$
6. $\neg e(b,a) \vee \neg e(a,c)$
7. $e(a,b)$
8. $e(a,b) \to e(b,a)$
9. $e(b,a)$
10. $\neg\neg e(b,a)$
11. $\neg e(a,c)$

10. Justify the steps in the following argument.

$\forall y \forall x[(L(x,y) \wedge L(y,x)) \to (x=y)]$
$\forall x \forall y \forall z([L(x,y) \wedge L(y,z)] \to L(x,z))$
$\forall x L(x,x)$
$\forall x L(a,x)$
$\forall x L(b,x)$

$\therefore a = b$

1. $\forall x L(a,x)$
2. $L(a,b)$
3. $\forall x L(b,x)$
4. $L(b,a)$
5. $L(a,b) \wedge L(b,a)$
6. $\forall y \forall x[(L(x,y) \wedge L(y,x)) \to (x=y)]$
7. $(L(a,b) \wedge L(b,a)) \to (a=b)]$

8. $a = b$

4 Chapter 3 - Set Theory

1. Let $A = \{n \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z}\ [n = 3k]\}$, $B = \{n \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z}\ [n = 6k]\}$, and $C = \{n \in \mathbb{Z} \mid \exists\, k \in \mathbb{Z}\ [n = 4k]\}$.

If the given statement is true, prove it. If it is false, provide a counterexample.

a. $A \subseteq B$.

b. $B \subseteq C$.

c. $\overline{A} \subseteq \overline{B}$.

d. $B \cap C = \emptyset$.

2. Let $A = \{1, \{1\}, \{2\}, 3\}$ and $B = \{1, 2, 3\}$. Determine whether the given statement is true or false. **Justify your answers.**

a. $2 \in B$.

b. $\{2\} \in B$.

c. $2 \subseteq B$.

d. $\{2\} \subset B$.

e. $\{2\} \subseteq A \cap B$.

f. $2 \in A \cup B$.

g. $\{2\} \subset \overline{\overline{A} \cup B}$.

h. $2 \in \overline{\overline{A} \cap B}$.

i. $\emptyset \in A$

j. $\emptyset \subseteq A$

k. $\emptyset \subset A$

3. Let A and B be sets. Prove the given statement or provide a counterexample:

a. $2^{A \cap B} = 2^A \cap 2^B$.

b. $2^{A \cup B} = 2^A \cup 2^B$.

4. Prove that if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.

5. Prove that if $C \subseteq A \cap B$, then $A \triangle B \subseteq \overline{C}$.

6. In each of the previous two problems, demonstrate the given statement with

a. A Venn diagram.

b. A membership table.

7. There are 25 students in a Math class. 21 attended class on Tuesday and 19 attended class on Thursday, while 2 did not show up either day. How many students attended both classes?

8. There are 25 students in a Math class. 24 attended class on Monday, 21 attended class on Tuesday and 19 attended class on Thursday. 5 students missed class on either Monday or Tuesday, 17 students were in class on Tuesday and Thursday, and 9 students missed either Monday or Thursday. Every student came to class at least once. How many students attended every class?

9. How many permutations of the 26 different letters of the alphabet contain:

a. either the pattern OUT or the pattern DIG?

b. neither the pattern MAN nor the pattern ANT?

10. How many arrangements of the letters in MISCELLANEOUS have no pair of consecutive identical letters?

5 Chapter 4 - Properties of the Integers, Induction

1. Let $f(x) = e^{2x}$, and recall that $f^{(n)}(x)$ denote the n^{th} **derivative** of f. For any $n \in \mathbb{N}$, let $S(n)$ be the statement:

$$S(n) \ : f^{(n)}(x) = 2^n e^{2x}$$

Prove the statement $\forall\ n \in \mathbb{N}\ S(n)$.

2. For any $n \in \mathbb{N}$, let $S(n)$ be the statement:

$$S(n) \ : \frac{d}{dx}(x^n) = nx^{n-1}$$

Prove the statement $\forall\ n \in \mathbb{N}\ S(n)$.

3. Recall that the Fibonacci sequence is defined by $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. For any $n \in \mathbb{N}$, let $S(n)$ be the statement:

$$S(n) \ : 2|f_{3n}$$

Prove the statement $\forall\ n \in \mathbb{N}\ S(n)$. [**Hint:** In the inductive step, show that $f_{3(n+1)} = 2f_{3n+1} + f_{3n}$.]

4. For any $n \in \mathbb{N}$, let $S(n)$ be the statement:

$$S(n) \ : 3^n < (n+1)!$$

Prove the statement $\forall\ n \geq 4\ S(n)$.

5. For any $n \in \mathbb{N}$, let $S(n)$ be the statement:

$$S(n) \ : 3^n < n!$$

Prove the statement $\forall\ n \geq 7\ S(n)$.

6. Prove the following statement or give a counterexample:

$$\forall\ x_1, \ldots, x_5 \in \mathbb{Z}\ [6|3x_1 + 2x_2 + 6x_3 - 18x_4 + 6x_5]$$

7. Prove the following statement or give a counterexample:

$$\forall\ x_1, \dots, x_5 \in \mathbb{Z}\ [4|4x_1 + 20x_2 - 12x_3 + 60x_4 - 100x_5]$$

8. Write the numbers 101 and 7495 in:

a. Binary

b. Octal

c. Decimal

d. Hexidecimal

9. Write the number 101_2 and 7495_{16} in:

a. Binary

b. Octal

c. Decimal

d. Hexidecimal

10. In each part, determine the quotient and remainder when b is divided by a.

a. $a = 12$, $b = 129$.

b. $a = 129$, $b = 12$.

c. $a = -12$, $b = 129$.

d. $a = 12$, $b = -129$.

e. $a = -12$, $b = -129$.

6 Chapter 5 - Relations and Functions

1. Let $A = \mathbb{Z}$, $B = \{a, b, c, d, e\}$, and $C = \mathbb{R}$. Determine whether each set below is a relation from A to B, A to C, or B to C. Also, determine if the set is a relation on A, B or C.

a. $\{(x, y) \mid x$ is the y^{th} letter of the alphabet $\}$

b. $\{(x, y) \mid y$ is the x^{th} letter of the alphabet $\}$

c. $\{(x, y) \mid x \leq y\}$

In each case, determine whether the given relation is a function. Justify you answer.

2. Write the relation

$$\mathcal{R} = \{(x, y) \in \mathbb{Z}^2 \mid xy = 5\}$$

in roster notation. Is it a map?

3. Prove that $A \times \emptyset = \emptyset$ for any set A.

4. In each part, what is $|A \times B|$? How many relations are there from A to B? How many relations on A are there? How maps maps from A to B are there?

a. $|A| = 17$, $|B| = 5$.

b. $|A| = 5$, $|B| = 11$.

5. Which of the following relations are maps? If it is a map, determine the domain, codomain, range, whether it is injective, surjective, bijective, or none of these.

a. $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{Z} \mid a \geq b\}$

b. $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{Z} \mid a = -b\}$

c. $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{Z} \mid a + b = 5\}$

d. $\mathcal{R} = \{(a, b) \in \mathbb{N} \times \mathbb{Z} \mid ab = 5\}$

6. If $|A| = 5$ and $|B| = 7$, how many injective maps from A to B are there? How many injective maps from B to A are there? How many surjective maps from A to B are there? How many surjective maps from B to A are there?

7. Alice, Bob, Carla, Daphne, Eric, Frances, and Gertrude are going on a trip. They will take three cars: a Rolls Royce, a Subaru, and a Toyota. How many ways can they go in the three cars?

8. Alice, Bob, Carla, Daphne, Eric, Frances, and Gertrude are going on a trip. They will take three cars: a Rolls Royce, a Subaru, and a Toyota. How many ways can they go in the three cars if Alice insists on driving her own car?

9. Once Alice, Bob, Carla, Daphne, Eric, Frances, and Gertrude get to their destination, they go out to dinner. The restaurant has limited seating, so they must sit at three (identical) tables. How many ways can they do this?

10. Which of the following binary operations are commutative? Which are associative? Which have an identity, and if so, what is it? Prove each affirmative answer and give a counterexample of each negative answer.

a. $f(a, b) = \frac{a}{b} + \frac{b}{a}$, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

b. $f(a, b) = \sqrt{ab}$, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$.

c. $f(a, b) = a^2 - 2ab + b^2$, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

11. **a.** Let $A \subseteq \mathbb{Z}$. Use the Pigeonhole Principle to prove that if $|A| \geq 3$, then A has two distinct elements whose sum is even. What is the map $f : A \to B$?

b. Let $f : A \times A \to B$ be a map, where $|B| = 4$. Find the minimal cardinality of A which ensures that there are distinct pairs $(x, y) \in A$ and $(w, z) \in A$ such that $f(x, y) = f(w, z)$.

c. Let $A \subseteq \mathbb{Z}$. Use your answer to the previous part to determine the minimal cardinality of A which ensures that there are distinct pairs $(x, y) \in A$ and $(w, z) \in A$ such that $x+w$ and $y+z$ are both even. [**Hint:** For any integer a, let $\bar{a}$ be the remainder upon division by 2 (so $\bar{a} = 0$ if a is even and $\bar{a} = 1$ if a is odd, let $B = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, and define $f(x, y) = (\bar{x}, \bar{y})$.]

12. Show that if any 14 integers are selected from the set $[25] = \{1, 2, \ldots, 25\}$, there are at least two whose sum is 26.

13. Prove that if $A \subseteq [100] = \{1, 2, \ldots, 100\}$, and $|A| \geq 11$, there are two distinct integers $x, y \in A$ such that $|\sqrt{x} - \sqrt{y}| < 1$.

14. Let $f = \{(1, 3), (2, 4), (3, 2), (4, 1)\}$, and let $g = \{(1, 1), (2, 3), (3, 2), (4, 1)\}$.

a. Find $g \circ f$ and $f \circ g$. Which of these are injective? Surjective? Justify your answers.

b. Find f^{-1}.

15. Let $A = 2^{[3]}$, and define maps $f : A \to A$ and $g : A \to A$ by $f(S) = S \cap \{1, 2\}$ and $g(S) = \overline{S}$. So, for example $f(\{1, 3\}) = \{1\}$ and $g(\{1, 3\}) = \{2\}$.

a. Explain why f is neither injective nor surjective.

b. Prove that g is bijective, and find the inverse.

7 Chapter 7 - Relations

1. Give an example of a relation on [4] which is:

a. reflexive and symmetric, but not transitive.

b. reflexive and transitive, but not symmetric.

c. symmetric and transitive, but not reflexive.

2. Determine whether each of the following relations are reflexive, symmetric, antisymmetric, and/or transitive. Justify your answers.

a. For a set S and a fixed subset $C \subseteq S$, the relation $A\mathcal{R}B$ holds if and only if $A \cap C = B \cap C$.

b. For a set S and a fixed subset $C \subseteq S$, the relation $A\mathcal{R}B$ holds if and only if $A \cup C = B \cup C$.

c. For a set S and a fixed subset $C \subseteq S$, the relation $A\mathcal{R}B$ holds if and only if $A \subseteq B$.

3. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be relations on a set A. Prove or give a counterexample of each of the following statements.

a. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are reflexive, so is $\mathcal{R}_1 \cap \mathcal{R}_2$.

b. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive, so is $\mathcal{R}_1 \cap \mathcal{R}_2$.

c. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are symmetric, so is $\mathcal{R}_1 \cap \mathcal{R}_2$.
[**Hint:** If we let $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$, then $a\mathcal{R}b$ if and only if $a\mathcal{R}_1b$ **and** $a\mathcal{R}_2b$.]

4. Draw the Hasse diagram for each of the following posets.

a. $([4], \leq)$

b. $(2^{[4]}, \subseteq)$

c. $(A, \prec)$, where $A = \{(x, y) \in [4] \times [4] \mid x < y\} = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$, and $(x, y) \prec (w, z)$ if and only if $x \leq w$ and $y \leq z$.

5. Fix $b \in \mathbb{Z}$, and define a relation on $\mathbb{Z}$ by $n\mathcal{R}m$ if and only if n and m have the same remainder upon division by b. Prove that $\mathcal{R}$ is an equivalence relation.

6. If $A = \{1, 2, 3, 4, 5\}$ and $\mathcal{R}$ is the equivalence relation that induces the partition $A = \{1, 2\} \cup \{3, 4\} \cup \{5\}$, what is $\mathcal{R}$?

7. Let $C = \{1, 3, 4\}$, and define an equivalence relation on $2^{[4]}$ by $A\mathcal{R}B$ if and only if $A \cap C = B \cap C$. Find the equivalence classes.