Chapter 2

Differential Calculus of Functions of One Variable

2.1 Functions and Limits

1. Each of the following conditions fails to define a function on any domain. State why.

   a. \( \sin f(x) = x \)

      Solution
      
      - If \( |x| > 1 \) this equation has no (real) solution for \( f(x) \).
      - If \( |x| \leq 1 \), each of the values
        
        \[ f(x) = \arcsin x + 2n\pi, \quad n \in \mathbb{Z} \quad \text{and} \quad f(x) = \pi - \arcsin x + 2n\pi, \quad n \in \mathbb{Z} \]

        will satisfy the given equation equation. Hence the assigned value \( f(x) \) is not unique and therefore \( f \) is not a function.

   b. \( e^{f(x)} = -|x| \)

      Solution
      
      Since the exponential function \( g(x) = e^x \) is strictly positive, the given equation has no solution for any value of \( x \).

   c. \( 1 + x^2 + [f(x)]^2 = 0 \)

      Solution
      
      Since \( 1 + x^2 + [f(x)]^2 \geq 1 \), the given equation has no solution for any value of \( x \).

   d. \( f(x)[f(x) - 1] = x^2 \)

      Solution
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This equation is equivalent to \( f^2(x) - f(x) - x^2 = 0 \). Using the quadratic formula, we obtain

\[
f(x) = \frac{1 \pm \sqrt{1 + 4x^2}}{2}
\]

Again, the assigned value \( f(x) \) is not unique and therefore \( f \) is not a function.

3. Find \( D_f \).

a. \( f(x) = \tan{x} \)

**Solution**

\( D_f = \mathbb{R} - \{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \} \)

d. \( f(x) = \frac{\sin{x}}{x} \)

**Solution**

\( D_f = \mathbb{R} - \{0\} \)

e. \( e^{[f(x)]^2} = x, f(x) \geq 0 \)

**Solution**

Since \( e^y \) attains all values greater than or equal to one and none less than one, the given equation has a solution if and only if \( x \geq 1 \). Hence \( D_f = [1, \infty) \). (of course this is commensurate with the equivalent equation \( f(x) = \sqrt{\ln{x}} \))

4. Find \( \lim_{x \to x_0} f(x) \) and justify your answers with an \( \epsilon-\delta \) proof.

a. \( \lim_{x \to 1} (x^2 + 2x + 1) = 4 \)

**Solution**

Let \( f(x) = x^2 + 2x + 1, L = 4 \), and \( \epsilon > 0 \). Consider

\[
|f(x) - L| = |(x^2 + 2x + 1) - 4| = |x^2 + 2x - 3| = |x + 3||x - 1|
\]

Let \( \delta \leq 1 \). Then for all \( x \) with \( 0 < |x - 1| < \delta \leq 1 \)

\[-1 \leq x - 1 \leq 1, \text{ so } 3 \leq x + 3 \leq 5\]

Hence,

\[
|f(x) - L| = |x + 3||x - 1| \leq 5|x - 1|
\]

Choose \( \delta = \min\{1, \frac{\epsilon}{5}\} \), then for all \( x \) with \( 0 < |x - 1| < \delta \)

\[
|f(x) - L| \leq 5|x - 1| < 5 \cdot \frac{\epsilon}{5} = \epsilon
\]

This completes the proof.
b. \( \lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x \to 2} (x^2 + 2x + 4) = 12 \)

**Solution**

Let \( f(x) = \frac{x^3 - 8}{x - 2}, L = 12, \) and \( \epsilon > 0. \) For \( x \neq 2 \)

\[
|f(x) - L| = \left| \frac{x^3 - 8}{x - 2} - 12 \right| = |x^2 + 2x - 8| = |x + 4||x - 2|
\]

Let \( \delta \leq 1. \) Then for all \( x \) with \( 0 < |x - 2| < \delta \)

\[ -1 \leq x - 2 \leq 1, \text{ so } 5 \leq x + 4 \leq 7 \]

Hence,

\[ |f(x) - L| = |x + 4||x - 2| < 7|x - 2| \]

Choose \( \delta = \min \{1, \frac{\epsilon}{7}\}, \) then for all \( x \) with \( 0 < |x - 2| < \delta \)

\[ |f(x) - L| \leq 7|x - 2| < 7 \cdot \frac{\epsilon}{7} = \epsilon \]

This completes the proof.

d. \( \lim_{x \to 4} \sqrt{x} = 2 \)

**Solution**

Let \( f(x) = \sqrt{x}, L = 2, \) and \( \epsilon > 0. \) Consider

\[
|f(x) - L| = |\sqrt{x} - 2| = \left| \frac{x - 4}{\sqrt{x} + 2} \right| = \frac{1}{\sqrt{x} + 2} |x - 4| \leq \frac{1}{2} |x - 4|
\]

Choose \( \delta = 2\epsilon, \) then for all \( x \in D_f \) with \( 0 < |x - 4| < \delta \)

\[ |f(x) - L| \leq \frac{1}{2} |x - 4| < \frac{1}{2} \cdot 2\epsilon = \epsilon \]

This completes the proof.

7. Find \( \lim_{x \to x_0^-} f(x) \) and \( \lim_{x \to x_0^+} f(x) \), if they exist. Use \( \epsilon, \delta \) proofs, where applicable, to justify your answers.

b. \( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|}, \quad x_0 = 0 \)

**Solution**
• \( \lim_{x \to 0^-} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|} \right) \)

Observe that
\[
\lim_{x \to 0^-} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|} \right) = \lim_{x \to 0^-} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \left( -\frac{1}{x} \right) \right) \\
= \lim_{x \to 0^-} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} - \sin \frac{1}{x} \right) \\
= \lim_{x \to 0^-} x \cos \frac{1}{x}
\]

Finally, we will now show that \( \lim_{x \to 0^-} x \cos \frac{1}{x} = 0 \). Let \( \epsilon > 0 \). Consider
\[
\left| \frac{1}{x} \left( x \cos \frac{1}{x} - 0 \right) = |x| \left| \cos \frac{1}{x} \right| \leq |x| \right|
\]

Choose \( \delta = \epsilon \), then for all \( x \) with \( -\delta < x < 0 \)
\[
\left| \frac{1}{x} \left( x \cos \frac{1}{x} - 0 \right) \leq |x| < \delta = \epsilon \right|
\]

We conclude that
\[
\lim_{x \to 0^-} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|} \right) = \lim_{x \to 0^-} x \cos \frac{1}{x} = 0
\]

• \( \lim_{x \to 0^+} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|} \right) \)

Above we proved that \( \lim_{x \to 0^-} x \cos \frac{1}{x} = 0 \). In a similar manner it can be shown that \( \lim_{x \to 0^+} x \cos \frac{1}{x} = 0 \), therefore
\[
\lim_{x \to 0^+} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|} \right) = \lim_{x \to 0^+} \left( x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{x} \right) \\
= \lim_{x \to 0^+} \left( x \cos \frac{1}{x} + 2 \sin \frac{1}{x} \right) \\
= \lim_{x \to 0^+} x \cos \frac{1}{x} + \lim_{x \to 0^+} 2 \sin \frac{1}{x} \\
= \lim_{x \to 0^+} 2 \sin \frac{1}{x}
\]

We now prove that \( \lim_{x \to 0^+} 2 \sin \frac{1}{x} \) does not exist. Let \( L \in \mathbb{R}, \delta > 0 \) and \( \epsilon_0 = \max \{|2 - L|, |2 + L|\} \). By the Archimedean property of \( \mathbb{R} \), \( \exists n \in \mathbb{N} \) such that
\[
0 < x_1 = \frac{1}{\frac{\pi}{2} + 2n\pi} < \delta \quad \text{and} \quad 0 < x_2 = \frac{1}{\frac{\pi}{2} + 2n\pi} < \delta
\]
Note that $|2 \sin \frac{1}{x_1} - L| = |2 - L|$ and $|2 \sin \frac{1}{x_2} - L| = |-2 - L| = |2 + L|$. Hence, for every $L \in \mathbb{R}$, there exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists an $x$ with $0 < x < \delta$, with $|2 \sin \frac{1}{x} - L| \geq \varepsilon_0$. This shows that $\lim_{x \to 0^+} 2 \sin \frac{1}{x}$ does not exist. Therefore

$\lim_{x \to 0^+} \left(x \cos \frac{1}{x} + \sin \frac{1}{x} + \sin \frac{1}{|x|}\right)$ does not exist.

c. $\frac{|x-1|}{x^2+x-2}$, $x_0 = 1$

**Solution**

Use Theorem 2.1.4 adapted for one-sided limits.

- $\lim_{x \to 1^-} \frac{|x-1|}{x^2+x-2} = \lim_{x \to 1^-} \left(-\frac{x-1}{x^2+x-2}\right) = \lim_{x \to 1^-} \left(-\frac{1}{x+2}\right) = -\frac{1}{3}$
- $\lim_{x \to 1^+} \frac{|x-1|}{x^2+x-2} = \lim_{x \to 1^+} \frac{x-1}{x^2+x-2} = \lim_{x \to 1^+} \frac{1}{x+2} = \frac{1}{3}$

8. Prove: If $h(x) \geq 0$ for $a < x < x_0$ and $\lim_{x \to x_0^-} h(x)$ exists, then $\lim_{x \to x_0^-} h(x) \geq 0$. Conclude from this that if $f_2(x) \geq f_1(x)$ for $a < x < x_0$, then

$\lim_{x \to x_0^-} f_2(x) \geq \lim_{x \to x_0^-} f_1(x)$

if both limits exist.

**Solution**

In class I suggested you use a proof by contradiction. The essential ingredient of the proof can be found in the proof of Theorem 2.1.4. Part 4.

Assume $\lim_{x \to x_0^-} h(x) = L < 0$. Then $\exists \delta > 0$ such that for all $x$ with $x_0 - \delta < x < x_0$

$|h(x) - L| < \frac{|L|}{2}$

This implies that

$-\frac{|L|}{2} < h(x) - L < \frac{|L|}{2}$

so

$-\frac{|L|}{2} < h(x) < \frac{|L|}{2}$ and thus $-\frac{3|L|}{2} < h(x) < \frac{|L|}{2}$

a contradiction with the fact that $h(x) \geq 0$ for $a < x < x_0$.

Finally, choosing $h(x) = f_2(x) - f_1(x)$ yields

$\lim_{x \to x_0^-} (f_2(x) - f_1(x)) \geq 0$ and thus $\lim_{x \to x_0^-} f_2(x) \geq \lim_{x \to x_0^-} f_1(x)$

Provided both limits exist.
15. Find \( \lim_{{x \to \infty}} f(x) \) if it exists, and justify your answer directly from Definition 2.1.7.

b. \( \sin \frac{x}{x^\alpha} \) \((\alpha > 0)\)

**Solution**

We will show that \( \lim_{{x \to \infty}} \sin \frac{x}{x^\alpha} = 0 \). Let \( \epsilon > 0 \). Observe that

\[
\left| \sin \frac{x}{x^\alpha} - 0 \right| = \frac{\sin x}{|x|^\alpha} \leq \frac{1}{|x|^\alpha}
\]

Choose \( \tau = (1/\epsilon)^{1/\alpha} \) then for all \( x > \tau \)

\[
\left| \sin \frac{x}{x^\alpha} - 0 \right| \leq \frac{1}{|x|^\alpha} < \frac{1}{\tau^\alpha} = \frac{1}{(1/\epsilon)^{1/\alpha}} = \epsilon
\]

This completes the proof.

f. \( e^{-x^2}e^{2x} \)

**Solution**

We will show that \( \lim_{{x \to \infty}} \left( e^{-x^2}e^{2x} \right) = 0 \). Let \( \epsilon > 0 \). Observe that for \( x > 4 \)

\[
\left| e^{-x^2}e^{2x} - 0 \right| = e^{-x^2+2x} = e^{-\frac{1}{2}x^2+2x}e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x^2}(x-4)e^{-\frac{1}{2}x^2} < e^{-\frac{1}{2}x^2}
\]

- Note that if \( \epsilon > 1 \), then \( e^{-\frac{1}{2}x^2} < \epsilon \). Hence, with \( \tau = 4 \) and \( x > \tau \)

\[
\left| e^{-x^2}e^{2x} - 0 \right| < e^{-\frac{1}{2}x^2} < \epsilon
\]

- In case \( 0 < \epsilon \leq 1 \), we quickly solve the equation

\[
e^{-\frac{1}{2}x^2} = \epsilon
\]

for \( x \), yielding \( x = \sqrt{-2\ln \epsilon} \). Choose \( \tau = \max \{ 4, \sqrt{-2\ln \epsilon} \} \) then for \( x > \tau \)

\[
\left| e^{-x^2}e^{2x} - 0 \right| < e^{-\frac{1}{2}x^2} < e^{-\frac{1}{2}x^2} \leq e^{-\frac{1}{2}(\sqrt{-2\ln \epsilon})^2} = e^{\ln \epsilon} = \epsilon
\]

22. Find

c. \( \lim_{{x \to x_0}} \frac{1}{(x-x_0)^{2k}} \), \( k \) is a positive integer.

**Solution**

Observe that in the extended reals

\[
\lim_{{x \to x_0-}} \frac{1}{(x-x_0)^{2k}} = \infty \quad \text{and} \quad \lim_{{x \to x_0+}} \frac{1}{(x-x_0)^{2k}} = \infty
\]
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so

\[ \lim_{x \to x_0} \frac{1}{(x - x_0)^{2k+1}} = \infty \]

d. \( \lim_{x \to x_0} \frac{1}{(x - x_0)^{2k+1}} \), \( k \) is a positive integer.

Solution

Observe that in the extended reals

\[ \lim_{x \to x_0^-} \frac{1}{(x - x_0)^{2k+1}} = -\infty \quad \text{and} \quad \lim_{x \to x_0^+} \frac{1}{(x - x_0)^{2k+1}} = \infty \]

so, even in the extended reals, the undirected limit

\[ \lim_{x \to x_0} \frac{1}{(x - x_0)^{2k+1}} \]

does not exist.

2.2 Continuity

2. Prove that a function \( f \) is continuous at \( x_0 \) if and only if

\[ \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0) \]

Solution

• If:

Let \( \epsilon > 0 \). Since \( \lim_{x \to x_0^-} f(x) = f(x_0) \) there exists a \( \delta_1 > 0 \) such that

\[ |f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 - \delta_1 < x \leq x_0 \]

Similarly, since \( \lim_{x \to x_0^+} f(x) = f(x_0) \), there exists a \( \delta_2 > 0 \) such that

\[ |f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 \leq x < x_0 + \delta_2 \]

Choose \( \delta = \min \{ \delta_1, \delta_2 \} \), then

\[ |f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta \]

This shows that \( f \) is continuous at \( x_0 \).
• Only if:
Let $\epsilon > 0$. Since $f$ is continuous at $x_0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

This implies that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 - \delta < x \leq x_0, \text{ and}$$

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad x_0 \leq x < x_0 + \delta$$

Hence $\lim_{x \to x_0^-} f(x) = f(x_0)$ and $\lim_{x \to x_0^+} f(x) = f(x_0)$.

3. Determine whether $f$ is continuous from the left or from the right at $x_0$.

c. $f(x) = \frac{1}{x} \quad (x_0 = 0)$

Solution
Since $f(0)$ is undefined, the function $f$ is neither continuous from the left at 0, nor continuous from the right at 0.

g. $f(x) = \begin{cases} 
\frac{x + |x|(1 + x)}{x} \sin \frac{1}{x} & x \neq 0 \\
1 & x = 0 \quad (x_0 = 0)
\end{cases}$

Solution
Observe

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x + |x|(1 + x)}{x} \sin \frac{1}{x} = \lim_{x \to 0^-} \frac{x - x(1 + x)}{x} \sin \frac{1}{x}$$

$$= \lim_{x \to 0^-} \left( -x \sin \frac{1}{x} \right) = 0 \neq 1 = f(0)$$

Hence, the function $f$ is not continuous from the left at 0.

Note: The answer in the back of the book is not correct.

Similarly

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x + |x|(1 + x)}{x} \sin \frac{1}{x}$$

$$= \lim_{x \to 0^+} \frac{x + x(1 + x)}{x} \sin \frac{1}{x} = \lim_{x \to 0^+} (2 + x) \sin \frac{1}{x}$$

This limit is undefined. So, $f$ is not continuous from the right at 0 either.
5. Let
\[ g(x) = \frac{\sqrt{x}}{x-1} \]
On which of the following intervals is \( f \) continuous according to definition 2.2.3:
\([0,1), (0,1), (0,1], [1,\infty), (1,\infty)\)?

**Solution**
\([0,1), (0,1), \text{ and } (1,\infty)\).

11. Prove that the function \( g(x) = \log x \) is continuous on \((0, \infty)\). Take the following properties as given.

(a) \( \lim_{x \to 1} g(x) = 0 \)
(b) \( g(x_1) + g(x_2) = g(x_1x_2) \) if \( x_1, x_2 > 0 \).

**Solution**

Let \( \epsilon > 0 \) and \( x_0 \in (0, \infty) \). In class we showed that property (b) is equivalent to
\[ g(x_1) - g(x_2) = g\left(\frac{x_1}{x_2}\right) \quad \text{if} \quad x_1, x_2 > 0 \]

Let \( x \in (0, \infty) \). Consider
\[ |g(x) - g(x_0)| = \left| g\left(\frac{x}{x_0}\right) \right| \]

Since \( \lim_{x \to 1} g(x) = 0 \), there exists a \( \delta_1 > 0 \), such that
\[ |g(u)| = |g(u) - 0| = |g(u) - g(1)| < \epsilon \quad \text{whenever} \quad |u - 1| < \delta_1 \]

Observe that
\[ \left| \frac{x}{x_0} - 1 \right| = \left| \frac{x - x_0}{x_0} \right| < \delta_1 \quad \text{whenever} \quad |x - x_0| < \delta_1 |x_0| \]

Choose \( \delta = \delta_1 |x_0| \) then, by letting \( \frac{x}{x_0} \) play the role of \( u \), we may conclude that
\[ |g(x) - g(x_0)| = \left| g\left(\frac{x}{x_0}\right) \right| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta \]

Therefore \( g \) is continuous at \( x_0 \), and since \( x_0 \) was chosen arbitrarily on \((0, \infty)\) this shows that \( g \) is continuous on \((0, \infty)\).
16. Let $|f|$ be the function whose value at each $x$ in $D_f$ is $|f(x)|$. Prove: If $f$ is continuous at $x_0$, then so is $|f|$. Is the converse true?

**Solution**

Let $\epsilon > 0$, then

$$||f|(x) - |f|(x_0)| = ||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)|$$

Moreover, since $f$ is continuous at $x_0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$, so

$$||f|(x) - |f|(x_0)| \leq |f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta$$

Hence, the function $|f|$ is continuous at $x_0$.

The converse is not true. Consider for instance the function

$$f(x) = \begin{cases} \frac{|x|}{x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

Then $|f|$ is continuous at 0, but $f$ is not.

20. (a) Let $f_1$ and $f_2$ be continuous at $x_0$ and define

$$F(x) = \max \{f_1(x), f_2(x)\}$$

Show that $F$ is continuous at $x_0$.

**Solution**

The key idea for this proof is to make a distinction between the case that $f_1(x_0) = f_2(x_0)$, and the case that $f_1(x_0) \neq f_2(x_0)$. Let $\epsilon > 0$.

- Case 1: $f_1(x_0) = f_2(x_0)$

  Note that in this case $F(x_0) = \max \{f_1(x_0), f_2(x_0)\} = f_1(x_0) = f_2(x_0)$. Since $f_1$ is continuous at $x_0$, there exists a $\delta_1$ such that

  $$|f_1(x) - F(x_0)| = |f_1(x) - f_1(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta_1$$

  Similarly, since $f_2$ is continuous at $x_0$, there exists a $\delta_2$ such that

  $$|f_2(x) - F(x_0)| = |f_2(x) - f_2(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta_2$$

  Choose $\delta = \min \{\delta_1, \delta_2\}$. Then, because $F(x)$ either equals either $f_1(x)$ or $f_2(x)$,

  $$|F(x) - F(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta$$

  This shows that $F$ is continuous at $x_0$. 
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Case 2: \( f_1(x_0) \neq f_2(x_0) \)

Since \( f_1 \) is continuous at \( x_0 \), there exists a \( \delta_1 \) such that

\[
|f_1(x) - f_1(x_0)| < \epsilon \quad \text{whenever} \quad |x-x_0| < \delta_1
\]

Similarly, since \( f_2 \) is continuous at \( x_0 \), there exists a \( \delta_2 \) such that

\[
|f_2(x) - f_2(x_0)| < \epsilon \quad \text{whenever} \quad |x-x_0| < \delta_2
\]

Choose \( \delta = \min \{\delta_1, \delta_2\} \). Without loss of generality, we may assume that \( f_1(x_0) > f_2(x_0) \). Take \( \epsilon < \frac{f_1(x_0)-f_2(x_0)}{2} \). This assures that whenever \( |x-x_0| < \delta \)

\[
|F(x) - F(x_0)| = |f_1(x) - f_1(x_0)| < \epsilon
\]

Again, this shows that \( F \) is continuous at \( x_0 \).

(b) Let \( f_1, f_2, \ldots, f_n \) be continuous at \( x_0 \) and define

\[
F(x) = \max \{f_1(x), f_2(x), \ldots, f_n(x)\}
\]

Show that \( F \) is continuous at \( x_0 \).

Solution

Use mathematical induction. Let \( P_n \) denote the proposition mentioned above. Part (a) of this exercise shows that \( P_2 \) is true. Let \( n \) denote any positive integer greater than or equal to 2 and assume that \( P_2, P_3, \ldots, P_n \) are all true. Let

\[
h(x) = \max \{f_1(x), f_2(x), \ldots, f_n(x)\}
\]

then \( h \) and \( f_{n+1} \) are both continuous at \( x_0 \) and therefore

\[
F(x) = \max \{f_1(x), f_2(x), \ldots, f_{n+1}(x)\} = \max \{h(x), f_{n+1}(x)\}
\]

is continuous at \( x_0 \). Hence \( P_{n+1} \) is true and by the principle of mathematical induction we may conclude that \( P_n \) is true for all positive integers greater than or equal to 2.

21. Find the domains of \( f \circ g \) and \( g \circ f \).

a. \( f(x) = \sqrt{x}, \ g(x) = 1-x^2 \)

Solution

\( D_f = [0, \infty) \) and \( D_g = (-\infty, \infty) \).

- Let \( T = [-1, 1] \). Then \( T \subset D_g \) and \( g(x) \in D_f \) whenever \( x \in T \). The set \( T \) is the domain of \( f \circ g \).
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• Let $T = [0, \infty)$. Then $T \subset \text{Dom}_f$ and $f(x) \in \text{Dom}_g$ whenever $x \in T$. The set $T$ is the domain of $g \circ f$.

c. $f(x) = \frac{1}{1-x^2}$, $g(x) = \cos x$

Solution

$\text{Dom}_f = \{x \mid x \neq -1, 1\}$ and $\text{Dom}_g = (-\infty, \infty)$.

• Let $T = \{x \mid x \neq n\pi, n \in \mathbb{Z}\}$. Then $T \subset \text{Dom}_g$ and $g(x) \in \text{Dom}_f$ whenever $x \in T$. The set $T$ is the domain of $f \circ g$.

• Let $T = \{x \mid x \neq -1, 1\}$. Then $T \subset \text{Dom}_f$ and $f(x) \in \text{Dom}_g$ whenever $x \in T$. The set $T$ is the domain of $g \circ f$.

23. Use Theorem 2.2.7 to find all points $x_0$ at which the following functions are continuous.

a. $\sqrt{1 - x^2}$

Solution

Let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Then $g$ is continuous at all real $x_0$, while $f$ is continuous at all $x_0 > 0$. We conclude that $f \circ g$ is continuous at all $x_0$ with $1 - x_0^2 > 0$; that is the set

$$\{x_0 \mid 1 - x_0^2 > 0\} = (-1, 1)$$

g. $(1 - \sin^2 x)^{-1/2}$

Solution

Let $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 1 - \sin^2 x = \cos^2 x$. Then $g$ is continuous at all real $x_0$, while $f$ is continuous at all $x_0 > 0$. We conclude that $f \circ g$ is continuous at all $x_0$ with $\cos^2 x_0 > 0$; that is the set

$$\{x_0 \mid \cos^2 x_0 > 0\} = \{x_0 \mid \cos^2 x_0 \neq 0\} = \{x_0 \mid x_0 \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$$

Note: The answer in the back of the book is not correct.

24. Complete the proof of Theorem 2.2.9 by showing that there is an $x_2 \in [a, b]$ such that $f(x_2) = \beta$.

Solution

Recall that $f$ is continuous on $[a, b]$ and $\beta = \sup_{x \in [a, b]} f(x) = \sup \{f(x) \mid x \in [a, b]\}$. Suppose there is no $x_2 \in [a, b]$ such that $f(x_2) = \beta$. Then $f(x) < \beta$ for all $x \in [a, b]$. Let $t \in [a, b]$, then $f(t) < \beta$, so

$$f(t) < \frac{f(t) + \beta}{2} < \beta$$
Moreover, since \( f \) is continuous at \( t \), there exists an open interval \( I_t \) containing \( t \), such that
\[
f(x) < \frac{f(t) + \beta}{2} \text{ for all } x \in I_t \cap [a,b]
\]
Note that \( \mathcal{H} = \{ I_t \mid t \in [a,b] \} \) is an open covering of \([a,b]\), and since \([a,b]\) is compact, \( \mathcal{H} \) can be reduced to a finite sub-cover, say
\[
\{ I_{t_i} \mid 1 \leq i \leq n \}
\]
Let
\[
\beta_1 = \max_{1 \leq i \leq n} \frac{f(t_i) + \beta}{2}
\]
Observe that \( \beta_1 < \beta \), and
\[
f(x) < \beta_1 \text{ for all } x \in [a,b]
\]
This implies that \( \beta_1 \) is an upper bound for the set \( V = \{ f(x) \mid x \in [a,b] \} \) which is less than \( \beta = \sup V \), a contradiction. We conclude that there must be an \( x_2 \in [a,b] \) such that \( f(x_2) = \beta \).

### 2.3 Differentiable Functions of One Variable

3. Use Lemma 2.3.2. to prove that if \( f'(x_0) > 0 \), there is a \( \delta > 0 \) such that
\[
f(x) < f(x_0) \text{ if } x_0 - \delta < x < x_0 \text{ and } f(x) > f(x_0) \text{ if } x_0 < x < x_0 + \delta
\]
**Solution**
Since \( \lim_{x \to x_0} E(x) = 0 \), there exists a \( \delta > 0 \) such that \( |E(x)| < f'(x_0) \) whenever \( 0 < |x - x_0| < \delta \). Therefore, if \( 0 < |x - x_0| < \delta \),
\[
f'(x_0) + E(x) > 0
\]
Moreover, since
\[
f(x) - f(x_0) = [f'(x_0) + E(x)](x - x_0)
\]
the desired result follows.

5. Find all derivatives of \( f(x) = x^{n-1}|x| \), where \( n \) is a positive integer.
**Solution**
Recall that
\[
\frac{d}{dx}|x| = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases}
\]
Using the product rule we find that if \( x \neq 0 \)
\[
- f'(x) = (n - 1) x^{n-2} |x| + x^{n-1} \frac{|x|}{x} = (n - 1) x^{n-2} |x| + x^{n-2} |x| = nx^{n-2} |x|,
\]
and similarly
\[
- f''(x) = n (n - 1) x^{n-3} |x|
\]
\[
\ldots
\]
\[
f^{(k)}(x) = n (n - 1) \ldots (n - k + 1) x^{n-k-1} |x| \text{ for } 1 \leq k \leq n - 1
\]
Observe that the result for \( f^{(k)}(x) \) is based on the premise that \( f^{(k-1)}(x) \) is of the form \( ax^b |x| \) where \( a \) and \( b \) are positive integers. The \((n-1)^{st}\) derivative of \( f \) is no longer of this form
\[
f^{(n-1)}(x) = n (n - 1) \ldots 2 |x| = n! |x|
\]
Hence, its derivative needs to be examined separately
\[
- f^{(n)}(x) = \begin{cases} 
-n! & \text{if } x < 0 \\
0 & \text{if } x > 0
\end{cases} , \text{ and }
\]
- \( f^{(k)}(x) = 0 \) if \( k > n \).

Next we examine \( f^{(k)}(0) \). Observe that for integers \( m \)
\[
\left( \frac{d}{dx} x^m |x| \right)_{x=0} = \begin{cases} 
\lim_{x \to 0} \frac{x^m |x| - 0}{x - 0} = \lim_{x \to 0} x^{m-1} |x| = 0 & \text{if } m \geq 1 \\
\text{undefined} & \text{if } m < 1
\end{cases}
\]
Combining the results for \( x = 0 \) and \( x \neq 0 \) we obtain
\[
f^{(k)}(x) = n (n - 1) \ldots (n - k + 1) x^{n-k-1} |x| \text{ for } 1 \leq k \leq n - 1
\]
Note: The answer in the back of the book is not correct.
\[
f^{(n)}(x) = \begin{cases} 
-n! & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
 n! & \text{if } x > 0
\end{cases} , \text{ and }
\]
- \( f^{(k)}(x) = \begin{cases} 
0 & \text{if } x \neq 0 \\
\text{undefined} & \text{if } x = 0
\end{cases} \text{ for } k \geq n + 1.
\]

10. Prove Theorem 2.3.4 (b).

If \( f \) and \( g \) are differentiable at \( x_0 \), then so is \( f - g \) and
\[
(f - g)'(x_0) = f'(x_0) - g'(x_0)
\]
Solution
2.3. DIFFERENTIABLE FUNCTIONS OF ONE VARIABLE

Observe

\[
\lim_{x \to x_0} \frac{(f - g)(x) - (f - g)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - g(x) - [f(x_0) - g(x_0)]}{x - x_0} = \lim_{x \to x_0} \left[ \frac{f(x) - f(x_0) - [g(x) - g(x_0)]}{x - x_0} \right] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) - g'(x_0)
\]

We conclude that \((f - g)'(x_0)\) exists and equals \(f'(x_0) - g'(x_0)\).

11. Prove Theorem 2.3.4 (d).

If \(f\) and \(g\) are differentiable at \(x_0\) and \(g(x_0) \neq 0\), then \(f/g\) is differentiable at \(x_0\) and

\[
\left( \frac{f}{g} \right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}
\]

Solution

Observe

\[
\lim_{x \to x_0} \frac{\left( \frac{f}{g} \right)(x) - \left( \frac{f}{g} \right)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x_0)g(x_0)} = \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x) - f(x_0)g(x_0) + f(x_0)g(x)}{(x - x_0)g(x_0)g(x_0)} = \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x_0)g(x)} - \lim_{x \to x_0} \frac{f(x_0)g(x_0) - f(x_0)g(x)}{g(x_0)g(x)} = \frac{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}}{\lim_{x \to x_0} g(x)g(x_0)}
\]

Since \(g\) is differentiable at \(x_0\), \(g\) is continuous at \(x_0\), so \(\lim_{x \to x_0} g(x) = g(x_0)\), therefore

\[
\lim_{x \to x_0} \frac{\left( \frac{f}{g} \right)(x) - \left( \frac{f}{g} \right)(x_0)}{x - x_0} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}
\]

which proves the stated result.
15. a. Show that \( f'_+ (a) = f' (a+) \) if both quantities exist.

**Solution**

In class I suggested you use the Mean Value Theorem. We will do just that. If \( f' (a+) \) exists, then \( f' \) exists on some open interval \((a, a + \delta)\). If the right hand derivative \( f'_+ (a) \) exists, then \( f' \) is continuous from the right at \( a \). Let \( x^* \in (a, a + \delta) \), then \( f' \) is continuous on \([a, x^*]\) and differentiable on \((a, x^*)\), so the Mean Value Theorem applies and there exists a \( c \in (a, x^*) \) such that

\[
\frac{f (x^*) - f (a)}{x^* - a} = f' (c)
\]

This implies

\[
f'_+ (a) = \lim_{x^* \to a^+} \frac{f (x^*) - f (a)}{x^* - a} = \lim_{x^* \to a^+} f' (c)
\]

Finally, note that because \( c \in (a, x^*) \), \( c \to a^+ \) if \( x^* \to a^+ \) and since \( \lim_{x \to a^+} f' (x) \) exists

\[
\lim_{x^* \to a^+} f' (c) = \lim_{x \to a^+} f' (x) = f' (a+)
\]

We conclude that \( f'_+ (a) = f' (a+) \).

b. Example 2.3.4 shows that \( f'_+ (a) \) may exist even if \( f' (a+) \) does not. Give an example where \( f' (a+) \) exists but \( f'_+ (a) \) does not.

**Solution**

Take a simple function which is not continuous from the right at \( a \). For instance

\[
f (x) = \begin{cases} 
1 & \text{if } x = 0 \\
x & \text{if } x > 0
\end{cases}
\]

then

\[
\lim_{x \to 0^+} \frac{f (x) - f (0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 1}{x} \text{ is undefined}
\]

while \( f' (0+) = 1 \).

c. Complete the following statement so it becomes a theorem, and prove the theorem: "If \( f' (a+) \) exists and \( f \) is ______ at \( a \), then \( f'_+ (a) = f' (a+) \)."

**Solution**

"If \( f' (a+) \) exists and \( f \) is continuous from the right at \( a \), then \( f'_+ (a) = f' (a+) \)."

**Proof**

If \( f' (a+) \) exists, then \( f' \) exists on some open interval \((a, a + \delta)\). Let \( x^* \in (a, a + \delta) \), then since \( f \) is continuous from the right at \( a \), \( f \) is continuous on \([a, x^*]\) and differentiable on \((a, x^*)\), so the Mean Value Theorem applies and there exists a \( c \in (a, x^*) \) such that

\[
\frac{f (x^*) - f (a)}{x^* - a} = f' (c)
\]
This implies
\[ \lim_{x^* \to a^+} \frac{f(x^*) - f(a)}{x^* - a} = \lim_{x^* \to a^+} f'(c) \]

Note that because \( c \in (a, x^*) \), \( c \to a^+ \) if \( x^* \to a^+ \) and since \( \lim_{x \to a^+} f'(x) \) exists
\[ \lim_{x^* \to a^+} f'(c) = \lim_{x \to a^+} f'(x) = f'(a^+) \]

We conclude that \( f'_*(a) \) exists and equals \( f'(a^+) \).

20. Let \( n \) be a positive integer and
\[ f(x) = \frac{\sin nx}{n \sin x}, \quad x \neq k\pi \quad (k \text{ is integer}). \]

a. Define \( f(k\pi) \) such that \( f \) is continuous at \( k\pi \).

Solution
Let
\[ f(k\pi) = \lim_{x \to k\pi} f(x) = \lim_{x \to k\pi} \frac{\sin nx}{n \sin x} = \lim_{x \to k\pi} \frac{n \cos nx}{n \cos x} \]
\[ = \lim_{x \to k\pi} \frac{\cos nx}{\cos x} = \frac{\cos nk\pi}{\cos k\pi} = \frac{(-1)^{nk}}{(-1)^k} = (-1)^{(n-1)k} \]

and redefine \( f \) as
\[ f(x) = \begin{cases} 
\frac{\sin nx}{n \sin x} & x \neq k\pi, k \in \mathbb{Z} \\
(-1)^{(n-1)k} & x = k\pi, k \in \mathbb{Z}
\end{cases} \]

b. Show that if \( \bar{x} \) is a local extreme point of \( f \), then
\[ |f(\bar{x})| = \left[ 1 + (n^2 - 1) \sin^2 \bar{x} \right]^{-1/2} \]

HINT: Express \( \sin nx \) and \( \cos nx \) in terms of \( f(x) \) and \( f'(x) \), and add their squares to obtain a useful identity.

Solution
First consider the case that \( \bar{x} \neq k\pi, k \in \mathbb{Z} \), then \( f'(\bar{x}) = 0 \). Observe that
\[ \sin n\bar{x} = nf(\bar{x}) \sin \bar{x} \]

and
\[ 0 = f'(\bar{x}) = \frac{n \cos n\bar{x}}{n \sin \bar{x}} - \frac{\sin n\bar{x} \cos \bar{x}}{n \sin^2 \bar{x}} \]
\[ = \frac{\cos n\bar{x}}{\sin \bar{x}} - f(\bar{x}) \frac{\cos \bar{x}}{\sin \bar{x}} \]
hence
\[\cos n\pi = f(\pi) \cos \pi\]
therefore
\[1 = \cos^2 n\pi + \sin^2 n\pi = [f(\pi) \cos \pi]^2 + [nf(\pi) \sin \pi]^2 = f^2(\pi) [\cos^2 \pi + n^2 \sin^2 \pi] = f^2(\pi) [1 + (n^2 - 1) \sin^2 \pi]\]
which implies that
\[|f(\pi)| = \left[1 + (n^2 - 1) \sin^2 \pi\right]^{-1/2}\]
Moreover, if \(k \in \mathbb{Z}\), \(|f(k\pi)| = \left|(-1)^{(n-1)k}\right| = 1\), so the formula
\[|f(\pi)| = \left[1 + (n^2 - 1) \sin^2 \pi\right]^{-1/2}\]
is true even if \(\pi = k\pi, k \in \mathbb{Z}\).

c. Show that \(|f(x)| \leq 1\) for all \(x\). For what values of \(x\) is equality attained?

**Solution**
For integer \(k\), let \(I\) denote the closed and bounded interval \([k\pi, (k+1)\pi]\). Then \(f\) is continuous on \(I\) and by the Extreme Value Theorem \(f\) must have a minimum \(m\) and a maximum \(M\) on \(I\). Therefore there exists a local extreme point \(\pi\) on \(I\) such that \(m = f(\pi)\). Hence, by the result of Part (b) \(|m| = |f(\pi)| \leq 1\). In a similar fashion we can prove that \(|M| \leq 1\). This means that for all \(x\) in \(I\)
\[\bigl[-1 \leq -|m| \leq m \leq f(x) \leq M \leq |M| \leq 1\bigl]
so
\[|f(x)| \leq 1\] for all \(x\) in \(I\).

Because \(k\) was an arbitrary integer this implies that \(|f(x)| \leq 1\) for all real \(x\). If \(n = 1\) equality is attained for all \(x \in \mathbb{R}\), and if \(n > 1\) equality is attained for \(x = k\pi, k \in \mathbb{Z}\).

## 2.4 L’Hospital’s Rule

In Exercises 2 - 40, find the indicated limits.

6. \(\lim_{x \to 0} \frac{\log(1+x)}{x}\)

**Solution**
\[
\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{1}{1+x} = 1
\]
7. \( \lim_{x \to \infty} e^x \sin e^{-x^2} \)

Solution

\[
\lim_{x \to \infty} e^x \sin e^{-x^2} = \lim_{x \to \infty} \frac{\sin e^{-x^2}}{e^{-x}} = \lim_{x \to \infty} \frac{-2xe^{-x^2} \cos e^{-x^2}}{-e^{-x}} = \lim_{x \to \infty} \frac{2xe^{-x^2}}{(2x-1)e^{x^2-x}} = \lim_{x \to \infty} \frac{2}{(2x-1)e^{x(x-1)}} = 0
\]

20. \( \lim_{x \to 0} (1 + x)^{1/x} \)

Solution

\[
\lim_{x \to 0} (1 + x)^{1/x} = e^{\lim_{x \to 0} \frac{\log(1+x)}{x}} = e^{\lim_{x \to 0} \frac{1}{x+1}} = e^1 = e
\]

23. \( \lim_{x \to a^+} x^\alpha \log x \)

Solution

Observe that if \( \alpha \leq 0 \), \( \lim_{x \to a^+} x^\alpha \log x = -\infty \). When \( \alpha > 0 \), then

\[
\lim_{x \to a^+} x^\alpha \log x = \lim_{x \to a^+} \frac{\log x}{x^{-\alpha}} = \lim_{x \to a^+} \frac{1}{x^{-\alpha} - \alpha x^{-\alpha-1}} = \lim_{x \to a^+} \frac{1}{\alpha x^\alpha} = 0
\]

26. \( \lim_{x \to 1^+} \left( \frac{x+1}{x-1} \right)^{\sqrt[2]{\pi-1}} \)

Solution

\[
\lim_{x \to 1^+} \left( \frac{x+1}{x-1} \right)^{\sqrt[2]{\pi-1}} = e^{\lim_{x \to 1^+} \left[ \sqrt[2]{\pi-1} \log \left( \frac{x+1}{x-1} \right) \right]}
\]

\[
= e^{\lim_{x \to 1^+} \left[ \sqrt[2]{\pi-1} \log \left( \frac{x+1}{x-1} \right) \right]}
\]

\[
= e^{\lim_{x \to 1^+} \left[ \sqrt[2]{\pi-1} \log \left( \frac{x+1}{x-1} \right) - \sqrt[2]{\pi-1} \log(x-1) \right]}
\]

\[
= e^{\lim_{x \to 1^+} \left[ 2\sqrt[2]{\pi-1} \log(x-1) \right]}
\]

\[
= e^{\lim_{x \to 1^+} \left[ 2\sqrt[2]{\pi-1} \right]}
\]

\[
= e^{\lim_{x \to 1^+} \left[ 2\sqrt[2]{\pi-1} \right]}
\]

\[
= e= 1
\]

2.5 Taylor’s Theorem

2. Suppose \( f^{(n+1)}(x_0) \) exists, and let \( T_n \) be the \( n^{th} \) Taylor polynomial of \( f \) about \( x_0 \). Show that the function

\[
E_n(x) = \begin{cases} \frac{f(x) - T_n(x)}{(x-x_0)} & \text{if } x \in D_f - \{x_0\} \\ 0 & \text{if } x = 0 \end{cases}
\]
CHAPTER 2. DIFFERENTIAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

Is differentiable at \( x_0 \) and find \( E'_n (x_0) \).

**Solution**

Observe

\[
\lim_{x \to x_0} \frac{E_n(x) - E_n(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{E_n(x) - 0}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}}
\]

Let \( T_{n+1}(x) \) denote the \((n + 1)^{st}\) Taylor polynomial of \( f \) about \( x_0 \). Then

\[
\lim_{x \to x_0} \frac{E_n(x) - E_n(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^{n+1}} = \lim_{x \to x_0} \frac{f(x) - T_{n+1}(x) + T_{n+1}(x) - T_n(x)}{(x - x_0)^{n+1}}
\]

\[
= 0 + \lim_{x \to x_0} \left[ \frac{1}{(x - x_0)^{n+1}} \left\{ \frac{f^{(n+1)}(x_0)}{(n+1)!}(x - x_0)^{n+1} \right\} \right]
\]

\[
= \frac{f^{(n+1)}(x_0)}{(n+1)!}
\]

We conclude that

\[
E'_n (x_0) = \frac{f^{(n+1)}(x_0)}{(n+1)!}
\]

4. a. Prove: if \( f''(x_0) \) exists, then

\[
\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0)
\]

**Solution**

Since \( f''(x_0) \) exists, the second Taylor polynomial \( T_2 \) of \( f \) about \( x_0 \) is defined. We will compare \( f(x_0 + h) \) to \( T_2(x_0 + h) \), and \( f(x_0 + h) \) to \( T_2(x_0 + h) \). Recall

\[
\lim_{x \to x_0} \frac{f(x) - T_2(x)}{(x - x_0)^2} = 0
\]

If in this result \( x \) is replaced by \( x_0 + h \) we obtain

\[
\lim_{h \to 0} \frac{f(x_0 + h) - T_2(x_0 + h)}{h^2} = 0
\]
and similarly
\[ \lim_{h \to 0} \frac{f(x_0 - h) - T_2(x_0 - h)}{h^2} = 0 \]

Now observe
\[
\begin{align*}
&\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} \\
&= \lim_{h \to 0} \frac{f(x_0 + h) - T_2(x_0 + h)}{h^2} + \lim_{h \to 0} \frac{T_2(x_0 + h) - 2f(x_0) + T_2(x_0 - h)}{h^2} \\
&\quad + \lim_{h \to 0} \frac{f(x_0 - h) - T_2(x_0 - h)}{h^2} \\
&= \lim_{h \to 0} \frac{T_2(x_0 + h) - 2f(x_0) + T_2(x_0 - h)}{h^2} \\
&= \lim_{h \to 0} \left[ \frac{f(x_0) + f'(x_0)h + 1/2f''(x_0)h^2 - 2f(x_0)}{h^2} \\
&\quad + \frac{f(x_0) - f'(x_0)h + 1/2f''(x_0)h^2}{h^2} \right] \\
&= \lim_{h \to 0} \frac{f''(x_0)h^2}{h^2} = \lim_{h \to 0} f''(x_0) = f''(x_0)
\end{align*}
\]

b. Prove or give a counter example: If the limit in Part a exists, then so does $f''(x_0)$ and they are equal.

**Solution**

Notice that the proof of Part a is based on the assumption that $f''(x_0)$ exists. Without that assumption $T_2$ is undefined and the proof falls apart. To generate a counterexample for the given statement we will look for a function $f$ that quadratically approaches $f(x_0)$ as $x$ approaches $x_0$ and for which $f''(x_0)$ is undefined. With $x_0 = 0$, the function $f(x) = x|x|$ satisfies those requirements. We now verify that this function truly is a counterexample.

- $f'(0) = \lim_{x \to 0} \frac{x|x| - 0}{x} = \lim_{x \to 0} |x| = 0$, and if $x \neq 0$, $f'(x) = |x| + \frac{x|x|}{x} = 2|x$. So
  \[ \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \to 0} \frac{2|x|}{x} \text{ is undefined} \]
  Therefore $f''(0)$ is undefined.

- Observe
  \[
  \begin{align*}
  &\lim_{h \to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} \\
  &= \lim_{h \to 0} \frac{h|h| - h|\hbar|}{h^2} = \lim_{h \to 0} \frac{0}{h^2} = \lim_{h \to 0} 0 = 0
  \end{align*}
  \]
This means that with \( x_0 = 0 \), \( \lim_{h \to 0} f(x_0 + h) - 2f(x_0) + f(x_0 - h) = 0 \).

8. a. Let

\[
h(x) = \sum_{r=0}^{n} \alpha_r (x - x_0)^r
\]

be a polynomial of degree \( \leq n \) such that

\[
\lim_{x \to x_0} \frac{h(x)}{(x - x_0)^n} = 0
\]

Show that \( \alpha_r = 0 \) for \( 0 \leq r \leq n \).

**Solution**

We use induction on \( n \). Let \( P_n \) denote the proposition above.

- First we verify that \( P_0 \) is true. Note that if \( n = 0 \), then \( h(x) = \alpha_0 \) and \( (x - x_0)^n = 1 \), so

\[
\lim_{x \to x_0} \alpha_0 = 0
\]

Hence, \( \alpha_0 = 0 \).

- Let \( k \) denote a nonnegative integer and let \( P_k \) be true. Additionally let

\[
h(x) = \sum_{r=0}^{k+1} \alpha_r (x - x_0)^r \quad \text{and} \quad \lim_{x \to x_0} \frac{h(x)}{(x - x_0)^{k+1}} = 0
\]

Then

\[
\lim_{x \to x_0} h(x) = \lim_{x \to x_0} \left[ \frac{h(x)}{(x - x_0)^{k+1}} (x - x_0)^{k+1} \right] = 0 \cdot 0 = 0
\]

and since \( h \) is a polynomial it is continuous everywhere, so

\[
h(x_0) = \lim_{x \to x_0} h(x) = 0
\]

which means \( \alpha_0 = 0 \). Thus, with \( m = r - 1 \)

\[
h(x) = \sum_{r=1}^{k+1} \alpha_r (x - x_0)^r = \sum_{m=0}^{k} \alpha_{m+1} (x - x_0)^{m+1}
\]

and

\[
0 = \lim_{x \to x_0} \frac{h(x)}{(x - x_0)^{k+1}} = \lim_{x \to x_0} \frac{\sum_{m=0}^{k} \alpha_{m+1} (x - x_0)^{m+1}}{(x - x_0)^{k+1}}
\]

\[
= \lim_{x \to x_0} \frac{\sum_{m=0}^{k} \alpha_{m+1} (x - x_0)^m}{(x - x_0)^k}
\]
Observe that the numerator in the last expression is a polynomial of degree \( \leq k \), so the induction assumption applies and we may conclude that in addition to the fact that \( \alpha_0 = 0 \), also \( \alpha_r = 0 \) for \( 1 \leq r \leq k + 1 \). Hence \( P_{k+1} \) is true. This completes the proof.

b. Suppose \( f \) is \( n \) times differentiable at \( x_0 \) and \( p = \sum_{r=0}^{n} \alpha_r (x - x_0)^r \) is a polynomial of degree \( \leq n \) such that

\[
\lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^n} = 0
\]

Show that

\[
\alpha_r = \frac{f^{(r)}(x_0)}{r!} \quad \text{if} \quad 0 \leq r \leq n
\]

that is, \( p = T_n \), the \( n \)-th Taylor polynomial of \( f \) about \( x_0 \).

**Solution**

Observe

\[
0 = \lim_{x \to x_0} \frac{f(x) - p(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} + \lim_{x \to x_0} \frac{T_n(x) - p(x)}{(x - x_0)^n}
\]

we conclude that

\[
\lim_{x \to x_0} \frac{T_n(x) - p(x)}{(x - x_0)^n} = 0
\]

and by Part a, this implies that \( T_n(x) - p(x) \) is identically equal to zero, so \( p = T_n \).

16. Find an upper bound for the magnitude of the error in the approximation.

b. \( \sqrt{1 + x} \approx 1 + \frac{x}{2}, |x| < \frac{1}{8} \)

**Solution**

Let \( f(x) = \sqrt{1 + x} \), then use Taylor’s theorem with \( n = 1 \) and \( x_0 = 0 \).

\[
\sqrt{1 + x} - \left(1 + \frac{x}{2}\right) = \frac{f^{(2)}(c)}{2!} x^2
\]

so

\[
\left|\sqrt{1 + x} - \left(1 + \frac{x}{2}\right)\right| = \left|\frac{f^{(2)}(c)}{2!} x^2\right| \leq \frac{1}{2} \left(\frac{1}{8}\right)^2 \left|f^{(2)}(c)\right| = \frac{1}{128} \left|f^{(2)}(c)\right|
\]

Next we estimate \( f^{(2)}(c) = \frac{1}{4(c+1)^{3/2}} \) for \( |c| < \frac{1}{8} \). Observe

\[
\left|f^{(2)}(c)\right| = \frac{1}{4(c+1)^{3/2}} \leq \frac{1}{4 \left(-\frac{1}{8} + 1\right)^{3/2}} = \frac{4}{49} \sqrt{14}
\]
Hence, an upper bound for the magnitude of the error is given by

\[
\frac{1}{128} \cdot \frac{4}{49} \sqrt{14} = \frac{1}{1568} \sqrt{14} \approx 2.3863 \times 10^{-3}
\]

**Note:** The answer in the back of the book is not correct.

d. \(\log x \approx (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}, |x - 1| < \frac{1}{64}\)

**Solution**

Let \(f(x) = \log x\), then use Taylor’s theorem with \(n = 3\) and \(x_0 = 1\).

\[
\log x - \left( (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \right) = \frac{f^{(4)}(c)}{4!} (x - 1)^4
\]

so

\[
\left| \log x - \left( (x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} \right) \right| = \left| \frac{f^{(4)}(c)}{4!} (x - 1)^4 \right| \leq \frac{1}{24} \left( \frac{1}{64} \right)^4 \left| f^{(4)}(c) \right| = \frac{1}{402653184} \left| f^{(4)}(c) \right|
\]

Next we estimate \(\left| f^{(4)}(c) \right| = \frac{6}{c^4}\) for \(|c - 1| < \frac{1}{64}\). Observe

\[
\left| f^{(4)}(c) \right| = \frac{6}{c^4} \leq \frac{6}{\left( 1 - \frac{1}{64} \right)^4} = \frac{33554432}{5250987}
\]

Hence, an upper bound for the magnitude of the error is given by

\[
\frac{1}{402653184} \cdot \frac{33554432}{5250987} = \frac{1}{63011844} \approx 1.587 \times 10^{-8}
\]